A Theory of Effects and Resources: Adjunction Models and Polarised Calculi

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Curry-Howard-Lambek correspondence

for

effectful computation

(monads, call-by-push-value)

and resource-aware computation

(linear logic, resource modalities)

Outline

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Curry-Howard-Lambek

- A correspondence between programming languages, proof systems and algebraic structures
- e.g. *types / formulae / objects*

  $$A, B ::= 1 \mid A \times B \mid A \to B \mid A + B$$

- foundations

Curry-Howard-Lambek

Phenomenon *Similar structures* appearing in programming, logic, and algebra

Explanation A *paradigm* of higher-order computation

Role of the paradigm

- What is the starting point? What do we agree upon?
- Let us formulate and share scientific questions
- Criteria of scientific validity
- Go further:
  focus, study in more details, challenge the paradigm

Computational Effects and Linear Logic

Similar structures appearing in programming and logic

Computational effects

- Moggi: decompose effectful computation as $A \to TB$
- T is a *monad* that encapsulates effects
- Levy: refine into *adjunction models* (call-by-push-value: CBPV)

Linear logic

- Girard: decompose intuitionistic implication as $!A \multimap B$
- ! is a *co-monad* that manages resources (e.g. complexity)
- Seely, Benton and others: refine into a *symmetric monoidal adjunction*

Computational Effects and Linear Logic

Similar structures appearing in programming and logic

- Monads and co-monads decomposed into adjunctions
- Both call-by-value and call-by-name computations each time
- Tell-tale sign in formalisms: presence of *stacks* (or applicative contexts, abstract machines, *defunctionalised continuations*, left-introduction rules…)

“Extending Curry-Howard-Lambek”?

- ~~Proofs are programs~~

Instead:

- Looking for paradigms to guide us
- Normal scientific process

In this article:

- Decomposition of CBPV into Linear CBPV, that distinguishes the effect adjunction from the resource adjunction.
- Interpretation as models for calculi of intuitionistic logic (**LJ**) and intuitionistic linear logic (**ILL**) via polarisation: a paradigm of evaluation order.

"Extending Curry-Howard-Lambek"?

Incremental presentation:

$$
\begin{tikzcd}
\mathbf{MLJ}_p^\eta \arrow[dr, Rightarrow] \arrow[dd] & & \mathbf{IMLL}_p^\eta \arrow[ll] \arrow[dl] \arrow[dd] \\
& \mathbf{IMELL}_p^\eta \arrow[dd] & \\
\mathrm{CBPV} = \mathbf{LJ}_p^\eta \arrow[dr, Rightarrow] & & \mathbf{IMALL}_p^\eta \arrow[ll] \arrow[dl] \\
& \text{Linear CBPV} = \mathbf{ILL}_p^\eta &
\end{tikzcd}
$$

LJ: intuitionistic logic
ILL: intuitionistic linear logic
$^\eta_p$: "with evaluation order"

M: multiplicatives ($\otimes$, $\multimap$, 1)
A: additives ($\oplus$, $\&$, 0, $\top$)
E: exponential ($!$)

Outline

Presheaf enrichments

Cartesian enrichment

- $\mathscr{V}$ cartesian category of *values*. $(P, Q, \Gamma \ldots \in \mathscr{V})$
- Category $\underline{\mathscr{C}}$ enriched on $\mathscr{V}$-presheaves: morphisms $A \to B$ under context $\Gamma \in \mathscr{V}$

$$\underline{\mathscr{C}}_\Gamma(A, B)$$

- Identity and composition under a context:

$$\mathrm{id}_X^{(\Gamma)} \in \underline{\mathscr{C}}_\Gamma(X, X) \qquad \circ_{X,Y,Z}^{(\Gamma)} : \underline{\mathscr{C}}_\Gamma(Y, Z) \times \underline{\mathscr{C}}_\Gamma(X, Y) \to \underline{\mathscr{C}}_\Gamma(X, Z)$$

- Ex: $\underline{\mathscr{V}}$: $\qquad \underline{\mathscr{V}}_\Gamma(P, Q) = \mathscr{V}(\Gamma \times P, Q)$
- Category of *stacks* $\underline{\mathscr{S}}$ $\qquad (N, M \ldots \in \underline{\mathscr{S}})$
given with an enriched adjunction $F \dashv G$:

$$\underline{\mathscr{S}}_\Gamma(FP, N) \cong \underline{\mathscr{V}}_\Gamma(P, GN)$$

Presheaf enrichments

Symmetric monoidal enrichment

- $\mathscr{L}$ symmetric monoidal category of *linear values*. ($P, Q, \Gamma \ldots \in \mathscr{L}$)
- Category $\underline{\mathscr{C}}$ enriched on $\mathscr{L}$-presheaves: morphisms $A \to B$ under context $\Gamma \in \mathscr{L}$
  (Day's convolution monoidal structure on presheaves)
- Identity and composition under a linear context:

$$\mathrm{id}_X \in \underline{\mathscr{C}}_I(X, X) \qquad \circ^{(\Gamma,\Gamma')}_{X,Y,Z} : \underline{\mathscr{C}}_{\Gamma'}(Y, Z) \times \underline{\mathscr{C}}_{\Gamma}(X, Y) \to \underline{\mathscr{C}}_{\Gamma\otimes\Gamma'}(X, Z)$$

- Ex: $\underline{\mathscr{L}}$: $\qquad \underline{\mathscr{L}}_\Gamma(P, Q) = \mathscr{L}(\Gamma \otimes P, Q)$
- Category of *stacks* $\underline{\mathcal{S}}$ $\quad$ ($N, M \ldots \in \underline{\mathcal{S}}$)
  given with an enriched adjunction $F \dashv G$:

$$\underline{\mathcal{S}}_\Gamma(FP, N) \cong \underline{\mathscr{L}}_\Gamma(P, GN)$$

Functions, sums, resource modalities

Function space

- $\mathscr{V}/\mathscr{L}$-powers:

$$\underline{\mathscr{S}}_{\Gamma}\left(M, P \twoheadrightarrow N\right) \cong \underline{\mathscr{S}}_{\Gamma\otimes P}(M, N)$$

- $\mathbf{MLJ}_p^\eta$ models: $\underline{\mathscr{V}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{\mathscr{S}}$ where $\underline{\mathscr{S}}$ has $\mathscr{V}$-powers
  $\twoheadrightarrow$ is $\to$
  Coincides with EC models (Egger, Møgelberg, Simpson)
- $\mathbf{IMLL}_p^\eta$ models: $\underline{\mathscr{L}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{\mathscr{S}}$ where $\underline{\mathscr{S}}$ has $\mathscr{L}$-powers
  $\twoheadrightarrow$ is $\multimap$
- Optionally, cartesian product on $\underline{\mathscr{S}}$

Functions, sums, resource modalities

Sums

- $\mathscr{V}/\mathscr{L}$ (linearly) distributive $(\Gamma \otimes (P+Q) \cong \Gamma \otimes P + \Gamma \otimes Q)$
- Enrich on *distributive presheaves*:

$$\underline{\mathscr{C}}_0(X,Y) \cong 1 \qquad \underline{\mathscr{C}}_{\Gamma+\Gamma'}(X,Y) \cong \underline{\mathscr{C}}_\Gamma(X,Y) \times \underline{\mathscr{C}}_{\Gamma'}(X,Y)$$

- $\mathbf{LJ}_p^\eta$ model: $\mathbf{MLJ}_p^\eta$ model +
  $\mathscr{V}$ distributive, $\underline{\mathcal{S}}$ cartesian with distributive hom-presheaves
  Coincides with Call-by-push-value adjunction models (Levy)
- $\mathbf{IMALL}_p^\eta$ model: $\mathbf{IMLL}_p^\eta$ model +
  $\mathscr{L}$ lin. distributive, $\underline{\mathcal{S}}$ cartesian with distributive hom-presheaves

Functions, sums, resource modalities

Resource modalities

- $\mathbf{IMELL}_p^\eta$ model: $\mathbf{IMLL}_p^\eta$ model $\underline{\mathscr{L}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{\mathscr{S}}$ +

  symmetric monoidal adjunction $\mathscr{V} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathscr{L}$ (resource modality)
  Examples: **IMELL** ($\mathscr{L}$ SMCC), dialogue categories with resource modalities (Melliès and Tabareau)
- Every resource modality $\mathscr{V} \underset{\longleftarrow}{\overset{L}{\longrightarrow}} \mathscr{L}$ enriches into an adjunction

$$\underline{\mathscr{V}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{\mathscr{L}}_L$$

  ($\underline{\mathscr{C}}_L$ defined by precomposing presheaves with L, i.e. $(\underline{\mathscr{C}}_L)_A = \underline{\mathscr{C}}_{LA}$)
- This yields an $\mathbf{MLJ}_p^\eta$ model by composing adjunctions

$$\underline{\mathscr{V}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{\mathscr{L}}_L \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{\mathscr{S}}_L$$

  and defining $P \to N \overset{\text{def}}{=} LP \multimap N$ (“Girard” translation)

Functions, sums, resource modalities

Linear Call-by-push-value

- $\mathbf{ILL}_p^\eta$ model: $\mathbf{IMALL}_p^\eta$ model + resource modality

  Ex: Melliès' dialogue categories/chiralities with a resource modality and co-products ("$\mathbf{LL}_p^\eta$" as $\mathbf{ILL}_p^\eta$+involutive negation); linear local store models (to investigate).
- $\mathbf{ILL}_p^\eta$ models give rise to $\mathbf{LJ}_p^\eta$ models by composing adjunctions as before

Outline

A paradigm of evaluation order

$$\text{let } x^A \text{ be } t \text{ in } u$$

- Lazy: u before t
- Strict: t before u
- Types are positive or negative: $A, B ::= P \mid N$
- Positive types: $P, Q ::= 1 \mid X^+ \mid A \otimes B \mid A \oplus B \mid 0 \mid !A$
  Evaluate strictly
  (Polarities do not match focusing properties)
- Negative types: $N, M ::= \top \mid X^- \mid A \,\&\, B \mid A \to B$
  Evaluate lazily

A paradigm of evaluation order

A thought experiment:

$$\mathsf{let}\ y^A\ \mathsf{be}\ t\ \mathsf{in}\ \mathsf{let}\ x^B\ \mathsf{be}\ u\ \mathsf{in}\ v \stackrel{?}{=} \mathsf{let}\ x^B\ \mathsf{be}\ (\mathsf{let}\ y^A\ \mathsf{be}\ t\ \mathsf{in}\ u)\ \mathsf{in}\ v$$

A	B	
lazy	lazy	$=$
strict	strict	$=$
lazy	strict	$=$
strict	lazy	$\neq$

A paradigm of evaluation order

ML Define lazy composition with thunking

```
let y = f x in h (fun () -> g y) ≠ h (fun () -> g (f x))
```

Haskell Use $! as strict composition

```
(\y->h (g y)) $! (f x) ≠ h (g $! (f x))
```

A paradigm of evaluation order

FoSSaCS 2014 (M.)

- Axiomatization of a non-associative composition
- Reflection theorem with adjunctions
- Difference with focusing ($\Downarrow \dashv \Uparrow$ vs. $\uparrow \dashv \downarrow$)

Polarised calculi

$\mathbf{LJ}_p^\eta$ and $\mathbf{ILL}_p^\eta$

- Abstract-machine-like calculi:
  - Computation as reduction of pairs <expressions, context>
  - Constructs are defined as solutions to their abstract-machine transitions
  - Evaluation order determined by the type
- λ-calculi (call-by-name, call-by-value or both) obtained with syntactic sugar
- Type systems are the sequent calculi **LJ** and **ILL**, however cut-elimination follows an order
- “Barendregt-style”: Generation lemma, Decidability of typing, Subject reduction, Confluence, and similar properties of the λ-calculus in the style of Barendregt

Polarised calculi

Intepretation

- Interpret sequents $\Gamma, A \vdash B$ into the sets:

$$\underline{\mathcal{S}}_{\Gamma^+}(FA^+, B^-) \cong \underline{\mathcal{V}}_{\Gamma^+}(A^+, GB^-)$$

- $^+$ and $^-$ add F and G wherever necessary
- If A is positive, interpret cut as

$$\underline{\mathcal{S}}_{\Gamma_2^+}(FA, B^-) \times \underline{\mathcal{S}}_{\Gamma_1^+}(FI, FA) \to \underline{\mathcal{S}}_{\Gamma_1^+ \otimes \Gamma_2^+}(FI, B^-)$$

  Generalises the Kleisli composition for the monad GF
- If A is negative, interpret cut as

$$\underline{\mathcal{V}}_{\Gamma_2^+}(GA, GB^-) \times \underline{\mathcal{V}}_{\Gamma_1^+}(I, GA) \to \underline{\mathcal{V}}_{\Gamma_1^+ \otimes \Gamma_2^+}(I, GB^-)$$

  Generalises the Kleisli composition for the co-monad FG

Polarised calculi

Structural rules

- Do not represent structural rules explicitly
- All at once:

$$\frac{\Gamma \vdash t : A \mid}{\Gamma' \vdash t[\sigma] : A \mid}\ (\sigma\vdash)$$

  σ: substitution of variables for variables (*renaming*, *exchange*, and also *weakening* and *contraction* if non-linear)
  (as was done for separation logic in Bob Atkey's PhD thesis)
- Not syntax-directed: Coherence theorem
- First time linear logic is treated in this way: σ allows weakening and contraction on types $!A$
  A quotient for structural rules ($!$ is delicate)

Depolarisation

Equivalence between three properties:

1. Cuts associate
2. Evaluation order is unimportant (unrestricted β-reduction)
3. The effect adjunction is idempotent

"Depolarisation"

Suggests new approaches to intuitionistic logic and linear logic (see the works of Melliès)

Outline

1. Linear Call-by-Push-Value: effect adjunction + resource adjunction
2. Decomposition of cartesian models into linear ones (Girard translation)
3. Simple and general technique for calculi (in Barendregt-style)
4. Connection with intuitionistic (linear) logic via polarisation
5. Characterisation of depolarisation

- Do not conflate effects and resources
- Not a naive & one-to-one correspondence between algebraic models and calculi
- Start with abstract-machine-like calculi (λ-calculus is syntactic sugar)
- Do not represent structural rules explicitly (terms should provide a quotient modulo structural rules/the monoidal laws)

Thank you

Questions?