Duploid situations in concurrent games

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Impossibility results in denotational semantics

- Classical logic (CCC + $0^{0^-} \cong \mathrm{Id}$)
- Untyped λ-calculus with sums (Bi-CCC + $U^U \cong U$)

$\Rightarrow$Preorders

Adjunction models (Call-by-push-value)

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- $A, B \ldots \in |\mathcal{D}|, \quad f, g \ldots \in \mathcal{D}(A, B)$
- $\odot : \mathcal{D}(B, C) \times \mathcal{D}(A, B) \to \mathcal{D}(A, C), \quad \mathrm{id}_A \in \mathcal{D}(A, A)$ neutral

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Associativity of composition vs. evaluation order

- $\exists N, \exists f, f \circledcirc^N \perp \neq \perp$ (lazy evaluation)
- $\exists P, \forall g, g \circledcirc^P \perp = \perp$ (strict evaluation)
- $f \circledcirc^N (g \circledcirc^P \perp) \neq \perp$
- $(f \circledcirc^N g) \circledcirc^P \perp = \perp$

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- $f \circledcirc^N (g \circledcirc^P \bot) \neq \bot$
- $(f \circledcirc^N g) \circledcirc^P \bot = \bot$ "Blass problem" (Abramsky, Melliès)

Deductive systems from adjunctions

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$$(f \odot^A g) \odot^N h = f \odot^A (g \odot^N h)$$

$$\left(\begin{array}{l} (f \bullet g) \bullet h = f \bullet (g \bullet h) \\ (f \circ g) \circ h = f \circ (g \circ h) \\ (f \bullet g) \circ h = f \bullet (g \circ h) \\ \text{cf. duplicial algebras (Loday)} \end{array} \right)$$

Duploids: a characterisation

In $\mathcal{D}$ a deductive system

- f is *linear* $\iff \forall g, h : (f \odot g) \odot h = f \odot (g \odot h)$
- h is *thunkable* $\iff \forall f, g : (f \odot g) \odot h = f \odot (g \odot h)$
- A is *positive* $\iff \forall B, \forall f \in \mathcal{D}(A, B), f$ is linear
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Definition

A duploid is a deductive system where:

1. every object has a polarity;
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$$\begin{cases} (f \bullet g) \bullet h = f \bullet (g \bullet h) \\ (f \circ g) \circ h = f \circ (g \circ h) \\ (f \bullet g) \circ h = f \bullet (g \circ h) \end{cases}$$

$$\begin{cases} \delta_P \in \mathcal{D}(P, \Uparrow P), \delta_P^* \in \mathcal{D}(\Uparrow P, P) \\ \delta_P^* \circ (\delta_P \bullet f) = f \\ \delta_P \bullet \delta_P^* = \mathrm{id}_{\Uparrow P} \end{cases}$$

$$\begin{cases} \omega_N \in \mathcal{D}(N, \Downarrow N), \omega_N^* \in \mathcal{D}(\Downarrow N, N) \\ (f \circ \omega_N^*) \bullet \omega_N = f \\ \omega_N \circ \omega_N^* = \mathrm{id}_{\Downarrow N} \end{cases}$$

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1. $\Downarrow \vdash \Uparrow : \mathcal{P} \to \mathcal{N}$
2. η is pointwise a split mono: $\rho_N \circ \eta_N = \mathrm{id}_N$
3. ε is pointwise a split epi: $\varepsilon_P \bullet \theta_P = \mathrm{id}_P$
4. ρ (*run*) and θ (*thunk*), as non-natural families, must satisfy:
   - **4.1** $\rho_{\Uparrow} = \Uparrow\varepsilon$ and $\theta_{\Downarrow} = \Downarrow\eta$
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Definition

- $f \in \mathcal{N}(N, M)$ is *linear* if $\rho_M \circ \Uparrow\Downarrow f = f \circ \rho_N$
- $f \in \mathcal{P}(P, Q)$ is *thunkable* if $\theta_Q \bullet f = \Downarrow\Uparrow f \bullet \theta_P$

Proposition: $\Uparrow \vdash_{(\theta,\rho)} \Downarrow : \mathcal{N}_l \to \mathcal{P}_t$

Duploid Situations

Theorem

Any duploid situation conservatively extends into a duploid satisfying:

$$\Downarrow f = \omega \circ f \circ \omega^* \quad , \quad \eta = \delta \bullet \omega \quad , \quad \rho = \omega^* \bullet \delta^*$$

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Consider the collage of the adjunction $\Downarrow \vdash \Uparrow : \mathcal{P} \to \mathcal{N}$:

$$\mathcal{D}(P, Q) \stackrel{\text{def}}{=} \mathcal{P}(P, Q)$$
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and extend it as follows

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Conjecture: $\mathcal{D}upl \sim \mathcal{DS} \simeq \mathcal{A}dj_{\text{eq}} \triangleleft \mathcal{A}dj$

CBN: alternating negative strategies between negative arenas

A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \qquad : \qquad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

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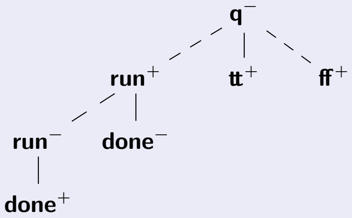

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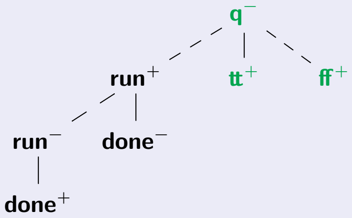

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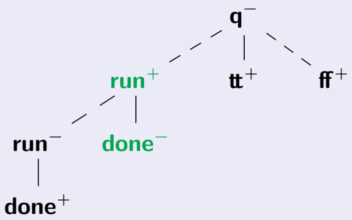

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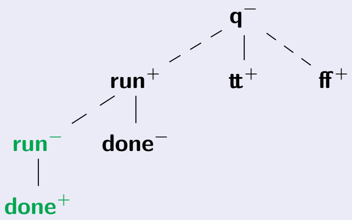

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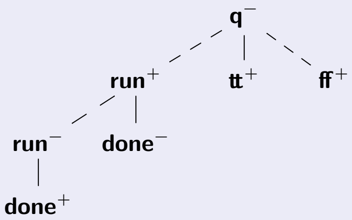

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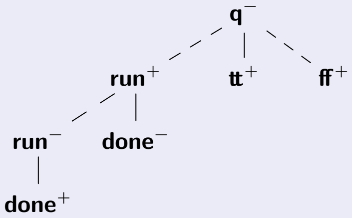

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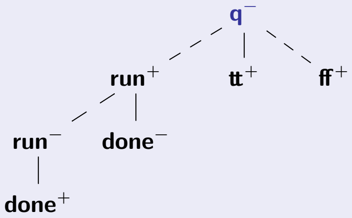

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$\mathbf{q}^-$

$\mathbf{run}^+$

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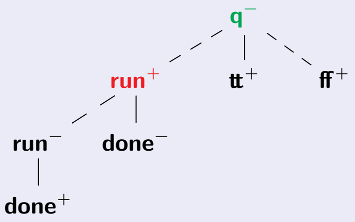

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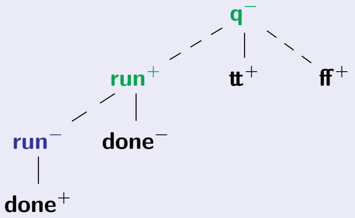

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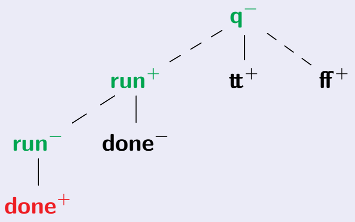

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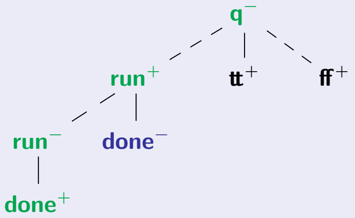

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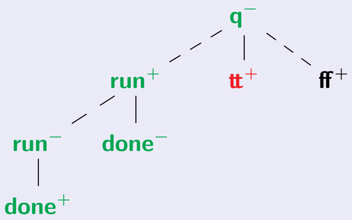
				$\mathbf{tt}^+$

An **arena**:



CBN: alternating negative strategies between negative arenas

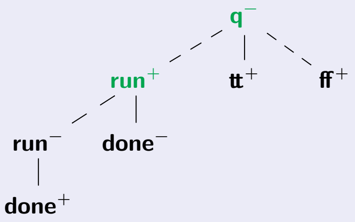
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A play:

$(\mathbf{com}$	$\to$	$\mathbf{com})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{run}^+$		

An arena:



CBN: alternating negative strategies between negative arenas

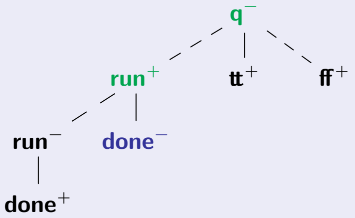
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:

$(\mathbf{com}$	$\to$	$\mathbf{com})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{run}^+$		
		$\mathbf{done}^-$		

An **arena**:



CBN: alternating negative strategies between negative arenas

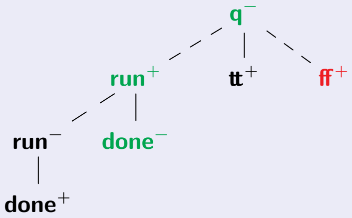
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \qquad : \qquad (\mathbf{com}\to\mathbf{com})\to\mathbb{B}$$

A **play**:

($\mathbf{com}$	$\to$	$\mathbf{com}$)	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{run}^+$		
		$\mathbf{done}^-$		
				$\mathbf{ff}^+$

An **arena**:

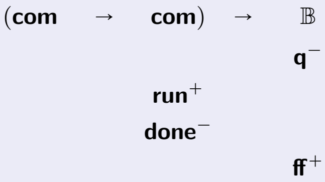


CBN: alternating negative strategies between negative arenas

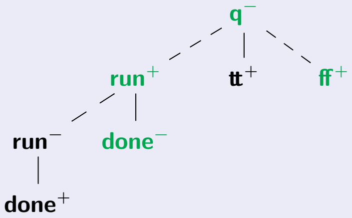
A **term**:

$$\lambda f^{\mathbf{com} \to \mathbf{com}}.\, \mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:



An **arena**:



CBN: alternating negative strategies between negative arenas

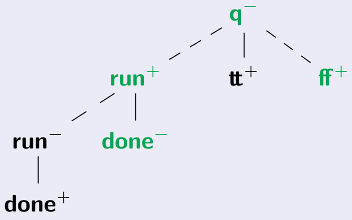
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:

$(\mathbf{com}$	$\to$	$\mathbf{com})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{run}^+$		
		$\mathbf{done}^-$		
				$\mathtt{ff}^+$

An **arena**:



Alternating plays: **alternating**, **linear orderings** of a **prefix** of the arena, **compatible** with the order of the arena.

CBN: alternating negative strategies between negative arenas

A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A play:

$$(\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

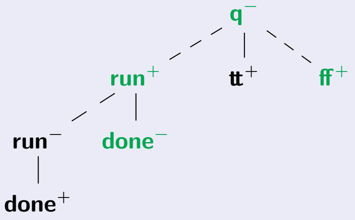
$\mathbf{q}^-$

$\mathbf{run}^+$

$\mathbf{done}^-$

$\mathbf{ff}^+$

An arena:



Alternating plays: **alternating**, **linear orderings** of a **prefix** of the arena, **compatible** with the order of the arena.

Negative, alternating strategy: certain sets of alternating plays.

CBV: alternating negative strategies between positive arenas

A **term**:

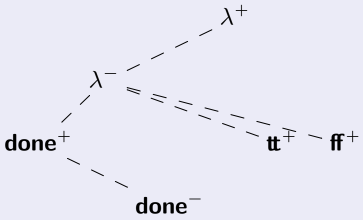
$$\lambda f^{\mathbf{com} \to \mathbf{com}}. \, \mathbf{newref} \; r \; \mathbf{in} \; f \; (r := \mathbf{tt}); \; !r \qquad : \qquad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com}\to\mathbf{com})\to\mathbb{B}$$

An **arena**:

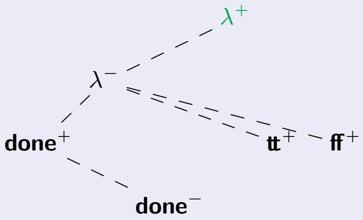


CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \qquad : \qquad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

An **arena**:

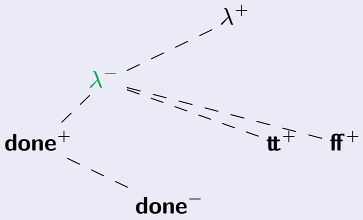


CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \qquad : \qquad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

An **arena**:

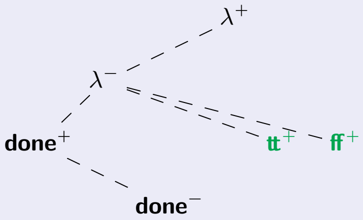


CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\mathbf{com} \to \mathbf{com}}.\, \mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

An **arena**:

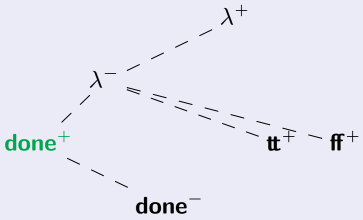


CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\mathbf{com} \to \mathbf{com}}.\, \mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\ \mathbf{com}\ \to \mathbf{com}) \to \mathbb{B}$$

An **arena**:

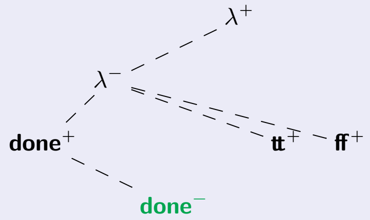


CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\, \mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

An **arena**:



CBV: alternating negative strategies between positive arenas

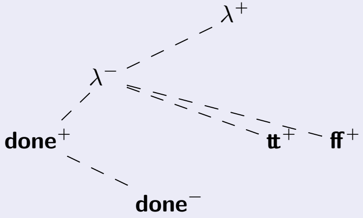
A **term**:

$$\lambda f^{\mathbf{com} \to \mathbf{com}}.\ \mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:

$$\vdash \quad (\mathbf{com} \quad \to \quad \mathbf{com}) \quad \to \quad \mathbb{B}$$

An **arena**:



CBV: alternating negative strategies between positive arenas

A **term**:

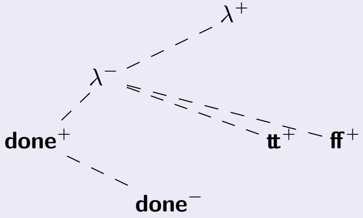
$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:

$$\vdash \quad (\mathbf{com} \quad \to \quad \mathbf{com}) \quad \to \quad \mathbb{B}$$

$\bullet^{-}$

An **arena**:



CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

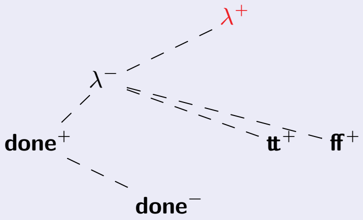
A **play**:

$$\vdash \quad (\mathbf{com} \quad \to \quad \mathbf{com}) \quad \to \quad \mathbb{B}$$

$\bullet^{-}$

λ^{+}

An **arena**:



CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\mathbf{com} \to \mathbf{com}}.\, \mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:

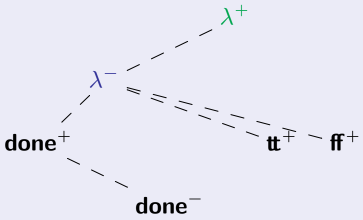
$\vdash \quad (\mathbf{com} \quad \to \quad \mathbf{com}) \quad \to \quad \mathbb{B}$

$\bullet^-$

λ^+

λ^-

An **arena**:



CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A play:

$\vdash \quad (\mathbf{com} \quad \to \quad \mathbf{com}) \quad \to \quad \mathbb{B}$

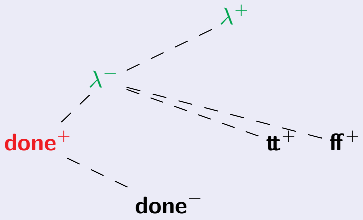
$\bullet^{-}$

λ^{+}

λ^{-}

$\mathbf{done}^{+}$

An arena:



CBV: alternating negative strategies between positive arenas

A **term**:

$$\lambda f^{\mathbf{com} \to \mathbf{com}}.\, \mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A play:

$\vdash \quad (\mathbf{com} \quad \to \quad \mathbf{com}) \quad \to \quad \mathbb{B}$

$\bullet^-$

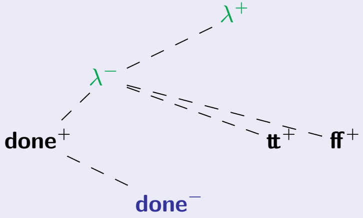
λ^+

λ^-

$\mathbf{done}^+$

$\mathbf{done}^-$

An arena:



CBV: alternating negative strategies between positive arenas

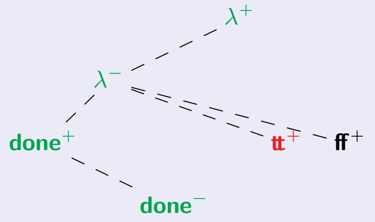
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:

	$\vdash$	$(\mathbf{com}$	$\to$	$\mathbf{com})$	$\to$	$\mathbb{B}$
$\bullet^-$						
					λ^+	
			λ^-			
		$\mathbf{done}^+$				
				$\mathbf{done}^-$		
						$\mathtt{tt}^+$

An **arena**:

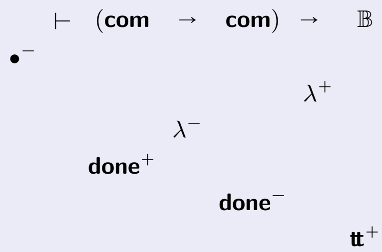


CBV: alternating negative strategies between positive arenas

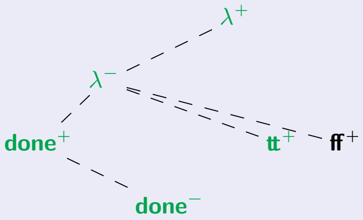
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A play:



An arena:

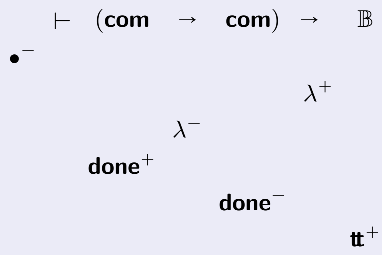


CBV: alternating negative strategies between positive arenas

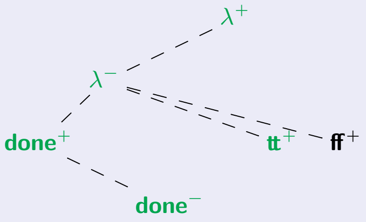
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A play:



An arena:



Negative, alternating strategies between **positive arenas**.

CBN, non-alternating

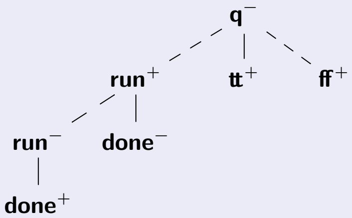
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com}\to\mathbf{com})\to\mathbb{B}$$

A **play**:

$$(\mathbf{com} \quad \to \quad \mathbf{com}) \quad \to \quad \mathbb{B}$$

An **arena**:



CBN, non-alternating

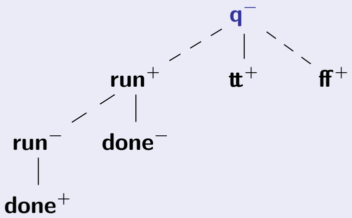
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:

(**com**	→	**com**)	→	$\mathbb{B}$
				$\mathbf{q}^-$

An **arena**:



CBN, non-alternating

A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

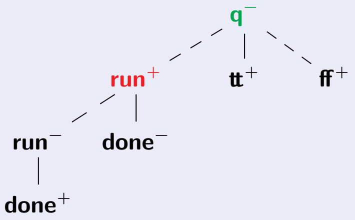
A play:

$(\mathbf{com} \quad \to \quad \mathbf{com}) \quad \to \quad \mathbb{B}$

$\mathbf{q}^-$

$\mathbf{run}^+$

An arena:



CBN, non-alternating

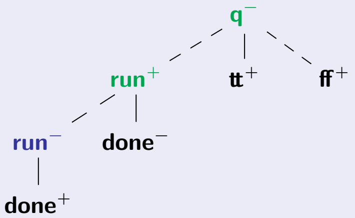
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \qquad : \qquad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:

$(\mathbf{com}$	$\to$	$\mathbf{com})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{run}^+$		
$\mathbf{run}^-$				

An **arena**:

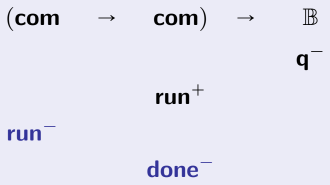


CBN, non-alternating

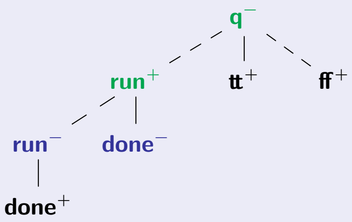
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:



An **arena**:



CBN, non-alternating

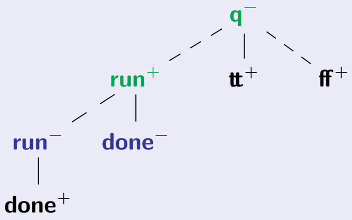
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A play:

$(\mathbf{com}$	$\to$	$\mathbf{com})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{run}^+$		
		$\mathbf{done}^-$		
$\mathbf{run}^-$				

An arena:

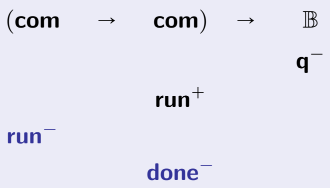


CBN, non-alternating

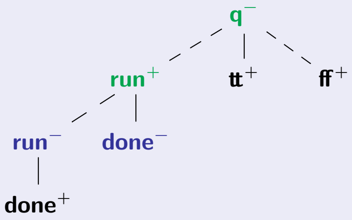
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathbf{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:



An **arena**:

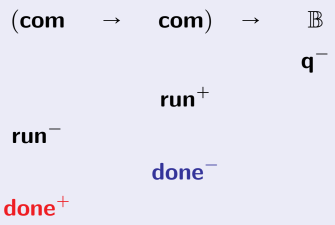


CBN, non-alternating

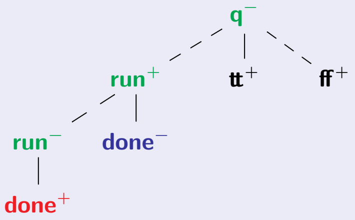
A **term**:

$$\lambda f^{\mathbf{com} \to \mathbf{com}}.\, \mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A play:



An arena:



CBN, non-alternating

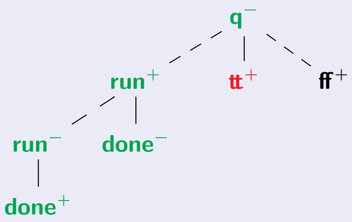
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A play:

$(\mathbf{com}$	$\to$	$\mathbf{com})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{run}^+$		
$\mathbf{run}^-$				
		$\mathbf{done}^-$		
$\mathbf{done}^+$				
				$\mathtt{tt}^+$

An arena:



CBN, non-alternating

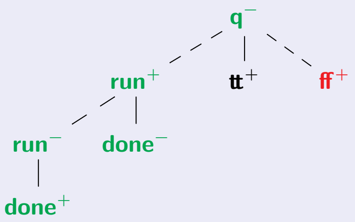
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A play:

$(\mathbf{com}$	$\to$	$\mathbf{com})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{run}^+$		
$\mathbf{run}^-$				
		$\mathbf{done}^-$		
$\mathbf{done}^+$				
				$\mathtt{ff}^+$

An arena:



CBN, non-alternating

A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:

$$(\mathbf{com} \quad \to \quad \mathbf{com}) \quad \to \quad \mathbb{B}$$

$\mathbf{q}^-$

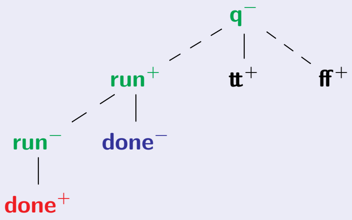
$\mathbf{run}^+$

$\mathbf{run}^-$

$\mathbf{done}^-$

$\mathbf{done}^+$

An **arena**:



CBN, non-alternating

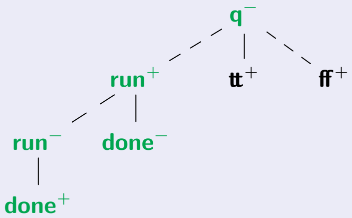
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:

$(\mathbf{com}$	$\to$	$\mathbf{com})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{run}^+$		
$\mathbf{run}^-$				
$\mathbf{done}^+$				
		$\mathbf{done}^-$		

An **arena**:



CBN, non-alternating

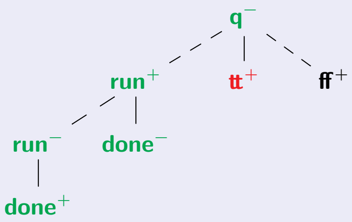
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com} \to \mathbf{com}) \to \mathbb{B}$$

A **play**:

(com	→	com)	→	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{run}^+$		
$\mathbf{run}^-$				
$\mathbf{done}^+$				
		$\mathbf{done}^-$		
				$\mathtt{tt}^+$

An **arena**:



CBN, non-alternating

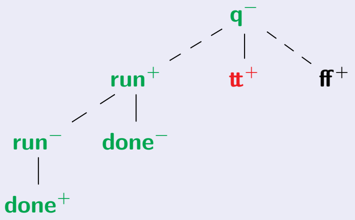
A **term**:

$$\lambda f^{\mathbf{com}\to\mathbf{com}}.\,\mathbf{newref}\ r\ \mathbf{in}\ f\ (r := \mathtt{tt});\ !r \quad : \quad (\mathbf{com}\to\mathbf{com})\to\mathbb{B}$$

A play:

$(\mathbf{com}$	$\to$	$\mathbf{com})$	$\to$	$\mathbb{B}$
				$\mathbf{q}^-$
		$\mathbf{run}^+$		
$\mathbf{run}^-$				
$\mathbf{done}^+$				
		$\mathbf{done}^-$		
				$\mathtt{tt}^+$

An arena:



Negative, non-alternating strategies: certain sets of non-alternating plays.

Duploids situations in toy categories of games

The **alternating world**:

$\mathcal{G}_{\text{alt}}^{-}$	$\mathcal{G}_{\text{alt}}^{+}$
neg. arenas	pos. arenas
neg. strat	neg. strats

Duploids situations in toy categories of games

The **alternating world**:

$$\mathcal{G}^{-}_{\mathrm{alt}} \overset{\Downarrow}{\longrightarrow} \mathcal{G}^{+}_{\mathrm{alt}}$$

neg. arenas	pos. arenas
neg. strat	neg. strats

Duploids situations in toy categories of games

The **alternating world**:

$$\mathcal{G}^{-}_{\text{alt}} \xrightarrow{\;\Downarrow\;} \mathcal{G}^{+}_{\text{alt}}$$

neg. arenas	pos. arenas
neg. strat	neg. strats

Down-shift on arenas

$\mathbb{B} =$

$$\mathbf{q}^{-}$$

$$\mathbf{tt}^{+} \qquad \mathbf{ff}^{+}$$

Duploids situations in toy categories of games

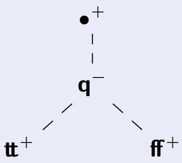
The **alternating world**:

$$\mathcal{G}^-_{\mathrm{alt}} \xrightarrow{\;\Downarrow\;} \mathcal{G}^+_{\mathrm{alt}}$$

neg. arenas	pos. arenas
neg. strat	neg. strats

Down-shift on arenas

$\Downarrow \mathbb{B} =$



Duploids situations in toy categories of games

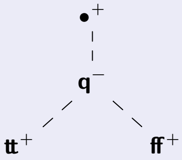
The **alternating world**:

$$\mathcal{G}^{-}_{\mathrm{alt}} \xrightarrow{\;\Downarrow\;} \mathcal{G}^{+}_{\mathrm{alt}}$$

neg. arenas	pos. arenas
neg. strat	neg. strats

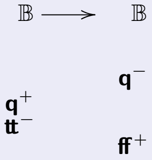
Down-shift on arenas

$\Downarrow \mathbb{B} =$



Down-shift on strategies

$\mathrm{neg} =$



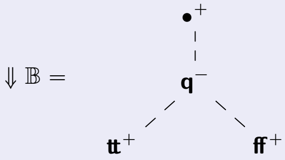
Duploids situations in toy categories of games

The **alternating world**:

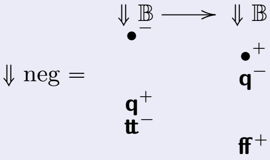
$$\mathcal{G}^{-}_{\mathrm{alt}} \xrightarrow{\Downarrow} \mathcal{G}^{+}_{\mathrm{alt}}$$

neg. arenas	pos. arenas
neg. strat	neg. strats

Down-shift on arenas



Down-shift on strategies



Duploids situations in toy categories of games

The **alternating world**:

$$\mathcal{G}^{-}_{\mathrm{alt}} \xrightarrow{\quad \Downarrow \quad} \mathcal{G}^{+}_{\mathrm{alt}}$$

neg. arenas	pos. arenas
neg. strat	neg. strats

Duploids situations in toy categories of games

The **alternating world**:

$$\mathcal{G}^{-}_{\mathrm{alt}} \underset{\Uparrow}{\overset{\Downarrow}{\rightleftarrows}} \mathcal{G}^{+}_{\mathrm{alt}}$$

neg. arenas	pos. arenas
neg. strat	neg. strats

Duploids situations in toy categories of games

The **alternating world**:

$$\mathcal{G}^{-}_{\mathrm{alt}} \underset{\Uparrow}{\overset{\Downarrow}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{G}^{+}_{\mathrm{alt}}$$

neg. arenas	pos. arenas
neg. strat	neg. strats

Duploids situations in toy categories of games

The **alternating world**:

$$\mathcal{G}^-_{\mathrm{alt}} \underset{\Uparrow}{\overset{\Downarrow}{\rightleftarrows}} \mathcal{G}^+_{\mathrm{alt}} \qquad (\bot)$$

neg. arenas — pos. arenas

neg. strat — neg. strats

Unit in $\mathcal{G}^-_{\mathrm{alt}}$ (natural)

$$\mathbb{B} \longrightarrow \Uparrow\Downarrow \mathbb{B}$$

$$\eta_{\mathbb{B}} = \quad \bullet^- \quad \bullet^+ \quad \mathbf{q}^- \quad \mathbf{q}^+ \quad \mathbf{tt}^- \quad \mathbf{tt}^+$$

Co-unit in $\mathcal{G}^+_{\mathrm{alt}}$ (natural)

$$\Downarrow\Uparrow \mathbb{B}^\bot \longrightarrow \mathbb{B}^\bot$$

$$\epsilon_{\mathbb{B}^\bot} = \quad \bullet^- \quad \bullet^+ \quad \mathbf{q}^- \quad \mathbf{q}^+ \quad \mathbf{tt}^- \quad \mathbf{tt}^+$$

Duploids situations in toy categories of games

The **alternating world**:

$$\mathcal{G}^-_{\mathrm{alt}} \underset{\Uparrow}{\overset{\Downarrow}{\rightleftarrows}} \mathcal{G}^+_{\mathrm{alt}} \qquad (\bot)$$

neg. arenas — pos. arenas

neg. strat — neg. strats

Unit in $\mathcal{G}^-_{\mathrm{alt}}$ (natural)

$$\eta_{\mathbb{B}} = \quad \mathbb{B} \longrightarrow \Uparrow\Downarrow\mathbb{B}$$

$\bullet^-$

$\bullet^+$

$\mathbf{q}^-$

$\mathbf{q}^+$

$\mathtt{tt}^-$

$\mathtt{tt}^+$

Co-unit in $\mathcal{G}^+_{\mathrm{alt}}$ (natural)

$$\epsilon_{\mathbb{B}^\bot} = \quad \Downarrow\Uparrow\mathbb{B}^\bot \longrightarrow \mathbb{B}^\bot$$

$\bullet^-$

$\bullet^+$

$\mathbf{q}^-$

$\mathbf{q}^+$

$\mathtt{tt}^-$

$\mathtt{tt}^+$

Variations at the heart of: work by Laurent (LLP), Levy (CBPV), Melliès (analysis of the Blass phenomenon).

Duploids situations in toy categories of games

The **alternating world**:

$$\mathcal{G}^{-}_{\mathrm{alt}} \underset{\Uparrow}{\overset{\Downarrow}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{G}^{+}_{\mathrm{alt}}$$

neg. arenas	pos. arenas
neg. strat	neg. strats

Duploids situations in toy categories of games

The **alternating world**:

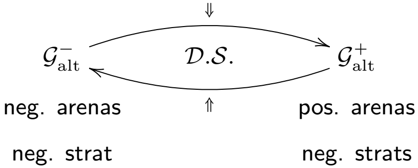
$$\mathcal{G}^{-}_{\mathrm{alt}} \underset{\Uparrow}{\overset{\Downarrow}{\rightleftarrows}} \mathcal{G}^{+}_{\mathrm{alt}} \qquad \mathcal{D}.\mathcal{S}.$$

neg. arenas — pos. arenas

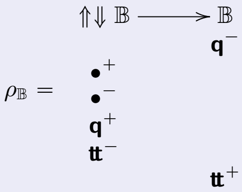
neg. strat — neg. strats

Duploids situations in toy categories of games

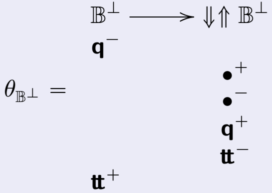
The **alternating world**:



Execution in $\mathcal{G}^-_{\mathrm{alt}}$ (not natural)



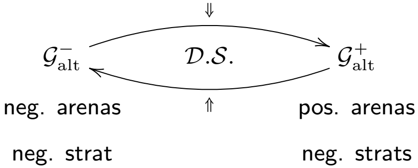
Thunking in $\mathcal{G}^+_{\mathrm{alt}}$ (not natural)



Naturality: no, except in the subcategories of **linear** and **thunkable** strategies.

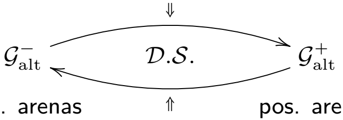
Duploids situations in toy categories of games

The **alternating world**:



Duploids situations in toy categories of games

The **alternating world**:



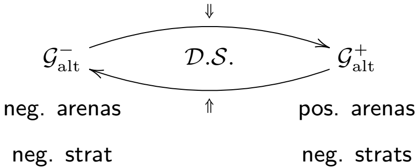
neg. arenas	pos. arenas
neg. strat	neg. strats

The **non-alternating world**:

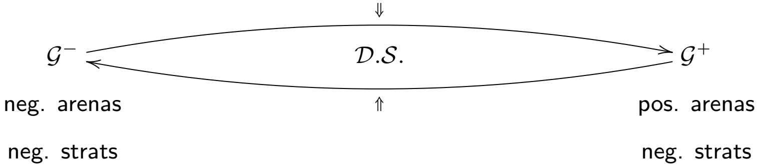
$\mathcal{G}^-$	$\mathcal{G}^+$
neg. arenas	pos. arenas
neg. strats	neg. strats

Duploids situations in toy categories of games

The **alternating world**:

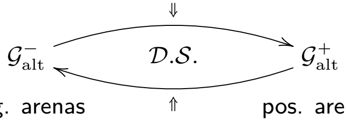


The **non-alternating world**:



Duploids situations in toy categories of games

The **alternating world**:



neg. arenas — pos. arenas

neg. strat — neg. strats

The **non-alternating world**:

$\mathcal{G}^-$ — $\mathcal{G}^+$

neg. arenas — pos. arenas

neg. strats — neg. strats

Duploids situations in toy categories of games

The **alternating world**:

$$\mathcal{G}^{-}_{\text{alt}} \underset{\Uparrow}{\overset{\Downarrow}{\rightleftarrows}} \mathcal{G}^{+}_{\text{alt}} \qquad \mathcal{D.S.}$$

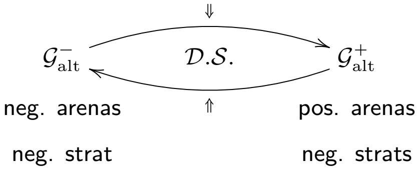
neg. arenas		pos. arenas
neg. strat		neg. strats

The **non-alternating world**:

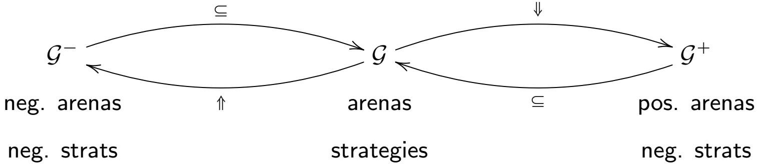
$\mathcal{G}^{-}$	$\mathcal{G}$	$\mathcal{G}^{+}$
neg. arenas	arenas	pos. arenas
neg. strats	strategies	neg. strats

Duploids situations in toy categories of games

The **alternating world**:

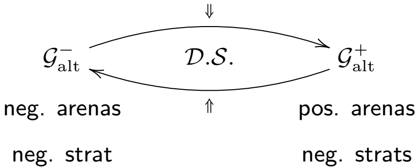


The **non-alternating world**:

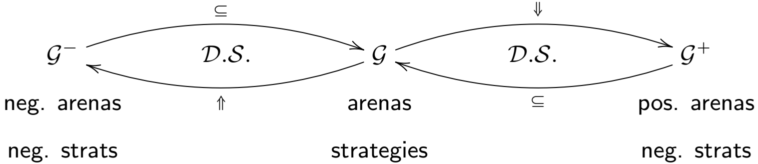


Duploids situations in toy categories of games

The **alternating world**:

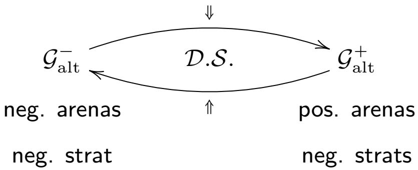


The **non-alternating world**:

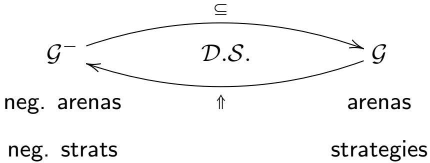


Duploids situations in toy categories of games

The **alternating world**:

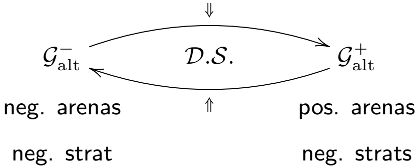


The **non-alternating world**:

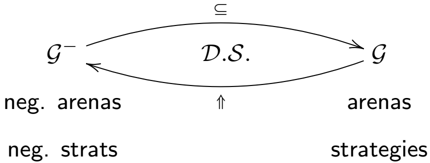


Duploids situations in toy categories of games

The **alternating world**:



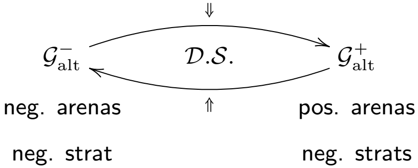
The **non-alternating world**:



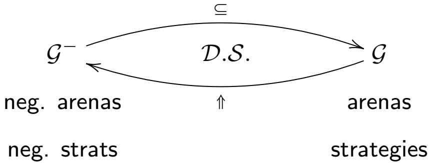
Note: This works in **non-alternating play-based games**, but also some **event-structure** based games (including edcs).

Duploids situations in toy categories of games

The **alternating world**:



The **non-alternating world**:



Note: This works in **non-alternating play-based games**, but also some **event-structure** based games (including edcs). **Not** in Conway games.

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G}$$

neg. arenas	all arenas
neg. strat	all strats

Duploids in non-alternating/concurrent games

$$\mathcal{G}^{-} \underset{\Uparrow}{\overset{\subseteq}{\underset{\perp}{\rightleftarrows}}} \mathcal{G}$$

neg. arenas	all arenas
neg. strat	all strats

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad (\bot)$$

neg. arenas	all arenas
neg. strat	all strats

Unit in $\mathcal{G}^-$ (natural)

$$N \longrightarrow \Uparrow N$$

$$\eta_N =$$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \quad (\bot)$$

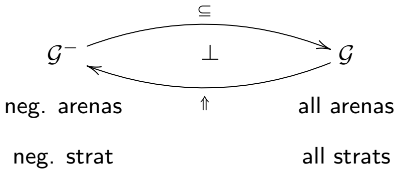
neg. arenas	all arenas
neg. strat	all strats

Unit in $\mathcal{G}^-$ (natural)

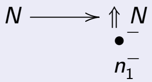
$$N \longrightarrow \Uparrow_{\bullet^-} N$$

$$\eta_N =$$

Duploids in non-alternating/concurrent games

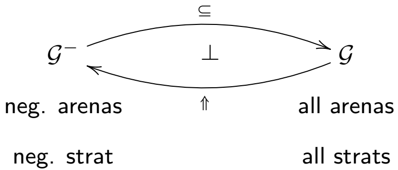


Unit in $\mathcal{G}^-$ (natural)

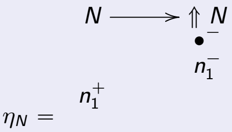


$\eta_N =$

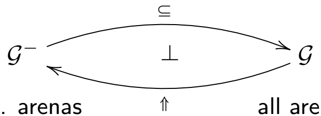
Duploids in non-alternating/concurrent games



Unit in $\mathcal{G}^-$ (natural)

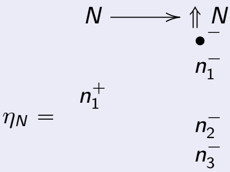


Duploids in non-alternating/concurrent games

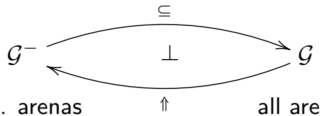


neg. arenas	all arenas
neg. strat	all strats

Unit in $\mathcal{G}^-$ (natural)



Duploids in non-alternating/concurrent games



neg. arenas — all arenas

neg. strat — all strats

Unit in $\mathcal{G}^-$ (natural)

$$N \longrightarrow \Uparrow N$$

$$\eta_N = \begin{array}{cc} & \bullet^- \\ & n_1^- \\ n_1^+ & \\ & n_2^- \\ & n_3^- \\ n_3^+ & \\ n_2^+ & \end{array}$$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \quad (\bot)$$

neg. arenas	all arenas
neg. strat	all strats

Unit in $\mathcal{G}^-$ (natural)

$$N \longrightarrow \Uparrow N$$

$$\eta_N = \quad \bullet^- \quad n_1^- \quad n_1^+ \quad n_2^- \quad n_3^- \quad n_3^+ \quad n_2^+$$

Co-unit in $\mathcal{G}$ (natural)

$$\Uparrow A \longrightarrow A$$

$$\epsilon_A =$$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad (\bot)$$

neg. arenas	all arenas
neg. strat	all strats

Unit in $\mathcal{G}^-$ (natural)

$$N \longrightarrow\ \Uparrow N$$

$$\eta_N = \quad \begin{matrix} & \bullet^- \\ & n_1^- \\ n_1^+ & \\ & n_2^- \\ & n_3^- \\ n_3^+ & \\ n_2^+ & \end{matrix}$$

Co-unit in $\mathcal{G}$ (natural)

$$\Uparrow A \longrightarrow A$$

$$\epsilon_A = \quad \begin{matrix} & a_1^- \end{matrix}$$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad (\bot)$$

neg. arenas	all arenas
neg. strat	all strats

Unit in $\mathcal{G}^-$ (natural)

$$N \longrightarrow\ \Uparrow N$$

$$\bullet^-$$

$$\eta_N = \quad n_1^+ \quad n_1^- \quad n_2^- \quad n_3^- \quad n_3^+ \quad n_2^+$$

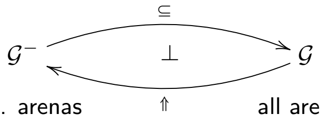
Co-unit in $\mathcal{G}$ (natural)

$$\Uparrow A \longrightarrow A$$

$$\bullet^+ \qquad a_1^-$$

$$\epsilon_A =$$

Duploids in non-alternating/concurrent games



neg. arenas	all arenas
neg. strat	all strats

Unit in $\mathcal{G}^-$ (natural)

$N \longrightarrow \Uparrow N$

$\bullet^-$

n_1^-

$\eta_N = \quad n_1^+$

n_2^-

n_3^-

n_3^+

n_2^+

Co-unit in $\mathcal{G}$ (natural)

$\Uparrow A \longrightarrow A$

a_1^-

$\bullet^+$

$\epsilon_A = \quad a_1^+$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad (\bot)$$

neg. arenas — all arenas

neg. strat — all strats

Unit in $\mathcal{G}^-$ (natural)

$N \longrightarrow \Uparrow N$

$$\eta_N = \quad \begin{array}{cc} & \bullet^- \\ & n_1^- \\ n_1^+ & \\ & n_2^- \\ & n_3^- \\ n_3^+ & \\ n_2^+ & \end{array}$$

Co-unit in $\mathcal{G}$ (natural)

$\Uparrow A \longrightarrow A$

$$\epsilon_A = \quad \begin{array}{cc} & a_1^- \\ \bullet^+ & \\ a_1^+ & \\ a_2^- & \\ & a_3^- \end{array}$$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad (\bot)$$

neg. arenas — all arenas

neg. strat — all strats

Unit in $\mathcal{G}^-$ (natural)

$$N \longrightarrow\ \Uparrow N$$

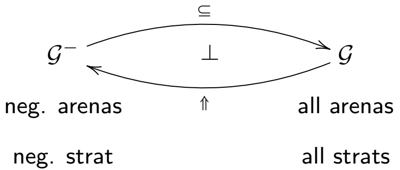
$$\eta_N = \quad \begin{matrix} & \bullet^- \\ & n_1^- \\ n_1^+ & \\ & n_2^- \\ & n_3^- \\ n_3^+ & \\ n_2^+ & \end{matrix}$$

Co-unit in $\mathcal{G}$ (natural)

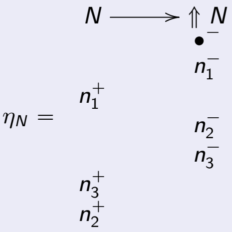
$$\Uparrow A \longrightarrow A$$

$$\epsilon_A = \quad \begin{matrix} & a_1^- \\ \bullet^+ & \\ a_1^+ & \\ a_2^- & \\ & a_3^- \\ a_3^+ & \end{matrix}$$

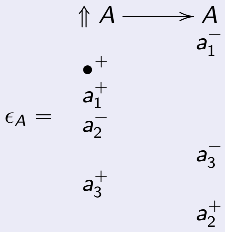
Duploids in non-alternating/concurrent games



Unit in $\mathcal{G}^-$ (natural)



Co-unit in $\mathcal{G}$ (natural)

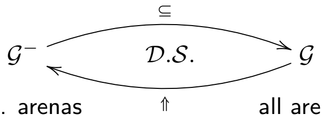


Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\underset{\bot}{\rightleftarrows}}} \mathcal{G}$$

neg. arenas	all arenas
neg. strat	all strats

Duploids in non-alternating/concurrent games



neg. arenas	all arenas
neg. strat	all strats

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D}.\mathcal{S}.$$

neg. arenas	all arenas
neg. strat	all strats

Execution in $\mathcal{G}^-$ (not natural)

$$\Uparrow N \longrightarrow N$$

$$\rho_N =$$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D}.\mathcal{S}.$$

neg. arenas	all arenas
neg. strat	all strats

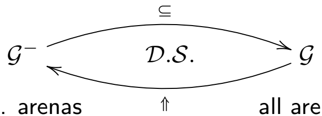
Execution in $\mathcal{G}^-$ (not natural)

$$\Uparrow N \longrightarrow N$$

$$n_1^-$$

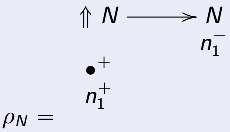
$\rho_N =$

Duploids in non-alternating/concurrent games



neg. arenas	all arenas
neg. strat	all strats

Execution in $\mathcal{G}^-$ (not natural)



Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D}.\mathcal{S}.$$

neg. arenas — all arenas

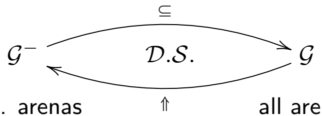
neg. strat — all strats

Execution in $\mathcal{G}^-$ (not natural)

$$\Uparrow N \longrightarrow N$$

$$\rho_N = \quad \bullet^+ \quad n_1^+ \quad n_3^- \qquad n_1^- \quad n_2^-$$

Duploids in non-alternating/concurrent games



neg. arenas — all arenas

neg. strat — all strats

Execution in $\mathcal{G}^-$ (not natural)

$\Uparrow N \longrightarrow N$

$$\rho_N = \begin{array}{cc} & n_1^- \\ \bullet^+ & \\ n_1^+ & \\ & n_2^- \\ n_3^- & \\ n_2^+ & \\ & n_3^+ \end{array}$$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D.S.}$$

neg. arenas — all arenas

neg. strat — all strats

Execution in $\mathcal{G}^-$ (not natural)

$$\Uparrow N \longrightarrow N$$

$$\rho_N = \quad \begin{array}{cc} & n_1^- \\ \bullet^+ & \\ n_1^+ & \\ & n_2^- \\ n_3^- & \\ n_2^+ & \\ & n_3^+ \end{array}$$

Thunking in $\mathcal{G}$ (not natural)

$$A \longrightarrow \Uparrow A$$

$$\theta_A =$$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D}.\mathcal{S}.$$

neg. arenas	all arenas
neg. strat	all strats

Execution in $\mathcal{G}^-$ (not natural)

$\Uparrow N \longrightarrow N$

$\rho_N =$

n_1^-

$\bullet^+$

n_1^+

n_2^-

n_3^-

n_2^+

n_3^+

Thunking in $\mathcal{G}$ (not natural)

$A \longrightarrow \Uparrow A$

a_1^-

$\theta_A =$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D.S.}$$

neg. arenas — all arenas

neg. strat — all strats

Execution in $\mathcal{G}^-$ (not natural)

$\Uparrow N \longrightarrow N$

$\rho_N =$ $\bullet^+$, n_1^+, n_3^-, n_2^+ (on $\Uparrow N$); n_1^-, n_2^-, n_3^+ (on N)

Thunking in $\mathcal{G}$ (not natural)

$A \longrightarrow \Uparrow A$

$\theta_A =$ a_1^- (on A); $\bullet^-$ (on $\Uparrow A$)

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D}.\mathcal{S}.$$

neg. arenas — all arenas

neg. strat — all strats

Execution in $\mathcal{G}^-$ (not natural)

$$\rho_N = \quad \Uparrow N \longrightarrow N$$

n_1^-

$\bullet^+$

n_1^+

n_2^-

n_3^-

n_2^+

n_3^+

Thunking in $\mathcal{G}$ (not natural)

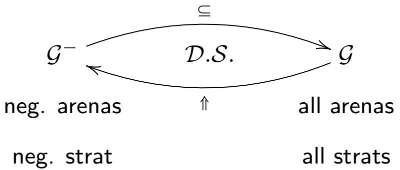
$$\theta_A = \quad A \longrightarrow \Uparrow A$$

a_1^-

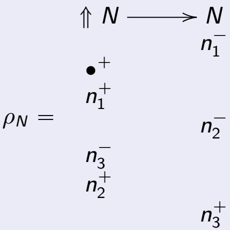
$\bullet^-$

a_1^+

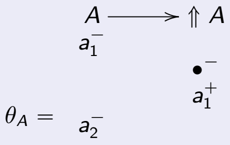
Duploids in non-alternating/concurrent games



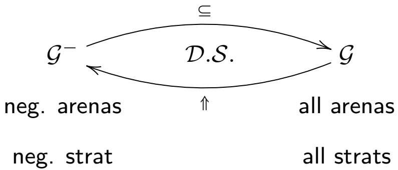
Execution in $\mathcal{G}^-$ (not natural)



Thunking in $\mathcal{G}$ (not natural)



Duploids in non-alternating/concurrent games



Execution in $\mathcal{G}^-$ (not natural)

$$\Uparrow N \longrightarrow N$$

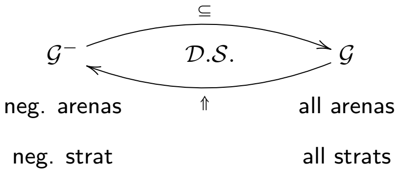
$$\rho_N = \quad \bullet^+ \quad n_1^+ \quad n_3^- \quad n_2^+ \qquad n_1^- \quad n_2^- \quad n_3^+$$

Thunking in $\mathcal{G}$ (not natural)

$$A \longrightarrow \Uparrow A$$

$$\theta_A = \quad a_1^- \quad a_2^- \qquad \bullet^- \quad a_1^+ \quad a_2^+$$

Duploids in non-alternating/concurrent games



Execution in $\mathcal{G}^-$ (not natural)

$\Uparrow N \longrightarrow N$

$\rho_N =$

n_1^-
$\bullet^+$
n_1^+
n_2^-
n_3^-
n_2^+
n_3^+

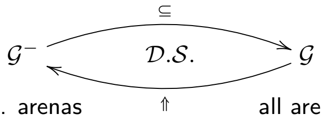
Thunking in $\mathcal{G}$ (not natural)

$A \longrightarrow \Uparrow A$

$\theta_A =$

a_1^-
$\bullet^-$
a_1^+
a_2^-
a_2^+
a_3^-

Duploids in non-alternating/concurrent games



neg. arenas — all arenas

neg. strat — all strats

Execution in $\mathcal{G}^-$ (not natural)

$\Uparrow N \longrightarrow N$

$\rho_N =$

$\Uparrow N$	N
	n_1^-
$\bullet^+$	
n_1^+	
	n_2^-
n_3^-	
n_2^+	
	n_3^+

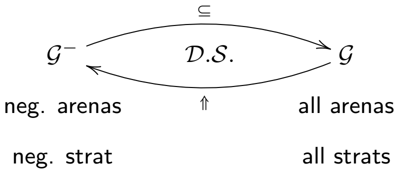
Thunking in $\mathcal{G}$ (not natural)

$A \longrightarrow \Uparrow A$

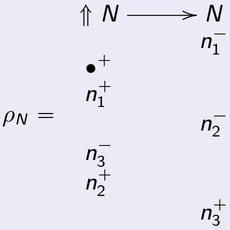
$\theta_A =$

A	$\Uparrow A$
a_1^-	
	$\bullet^-$
	a_1^+
a_2^-	
	a_2^+
	a_3^-
a_3^+	

Duploids in non-alternating/concurrent games



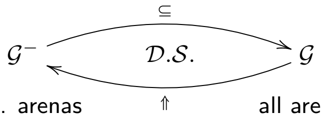
Execution in $\mathcal{G}^-$ (not natural)



Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

Duploids in non-alternating/concurrent games



neg. arenas | all arenas

neg. strat | all strats

Execution in $\mathcal{G}^-$ (not natural)

$\Uparrow N \longrightarrow N$

$\rho_N =$

$\Uparrow N$	N
	n_1^-
$\bullet^+$	
n_1^+	
	n_2^-
n_3^-	
n_2^+	
	n_3^+

Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

$\Leftrightarrow$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad (\mathcal{D}.\mathcal{S}.)$$

neg. arenas — all arenas

neg. strat — all strats

Execution in $\mathcal{G}^-$ (not natural)

$\Uparrow N \longrightarrow N$

$$\rho_N = \begin{array}{cc} & n_1^- \\ \bullet^+ & \\ n_1^+ & \\ & n_2^- \\ n_3^- & \\ n_2^+ & \\ & n_3^+ \end{array}$$

Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

$\Leftrightarrow$

$M \longrightarrow N$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D.S.}$$

neg. arenas	all arenas
neg. strat	all strats

Execution in $\mathcal{G}^-$ (not natural)

$$\rho_N = \begin{array}{ccc} \Uparrow N & \longrightarrow & N \\ & & n_1^- \\ \bullet^+ & & \\ n_1^+ & & \\ & & n_2^- \\ n_3^- & & \\ n_2^+ & & \\ & & n_3^+ \end{array}$$

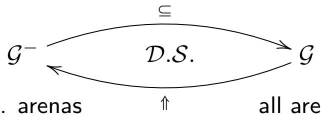
Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

$$\Leftrightarrow$$

$$\begin{array}{ccc} M & \longrightarrow & N \\ & & n_1 \end{array}$$

Duploids in non-alternating/concurrent games



neg. arenas	all arenas
neg. strat	all strats

Execution in $\mathcal{G}^-$ (not natural)

$$\rho_N = \begin{array}{ccc} \Uparrow N & \longrightarrow & N \\ & & n_1^- \\ \bullet^+ & & \\ n_1^+ & & \\ & & n_2^- \\ n_3^- & & \\ n_2^+ & & \\ & & n_3^+ \end{array}$$

Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

$$\Leftrightarrow$$

$$\begin{array}{ccc} M & \longrightarrow & N \\ & & n_1 \\ & & \cdots \\ & & n_k \end{array}$$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D}.\mathcal{S}.$$

neg. arenas — all arenas

neg. strat — all strats

Execution in $\mathcal{G}^-$ (not natural)

$\rho_N = \quad \Uparrow N \longrightarrow N$

n_1^-

$\bullet^+$

n_1^+

n_2^-

n_3^-

n_2^+

n_3^+

Proposition

A strategy $\sigma \in \mathcal{G}^-(M, N)$ is **linear** if it commutes with ρ

$\Leftrightarrow$

$M \longrightarrow N$

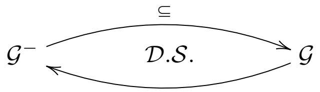
n_1

$\dots$

n_k

m^+

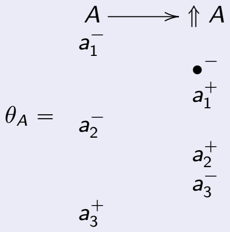
Duploids in non-alternating/concurrent games



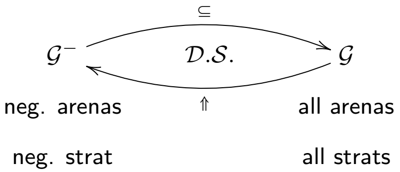
neg. arenas | all arenas

neg. strat | all strats

Thunking in $\mathcal{G}$ (not natural)



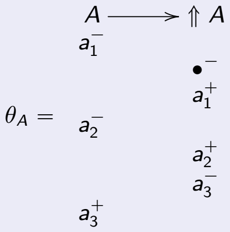
Duploids in non-alternating/concurrent games



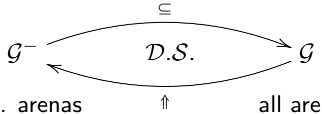
Proposition

A strategy $\sigma : A \to B$ is **thunkable** if it commutes with θ

Thunking in $\mathcal{G}$ (not natural)



Duploids in non-alternating/concurrent games



neg. arenas — all arenas

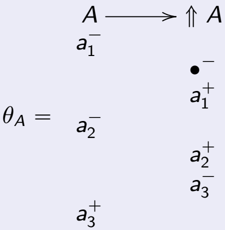
neg. strat — all strats

Proposition

A strategy $\sigma : A \to B$ is **thunkable** if it commutes with θ

$$\Leftrightarrow$$

Thunking in $\mathcal{G}$ (not natural)



Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D}.\mathcal{S}.$$

neg. arenas		all arenas
neg. strat		all strats

Proposition

A strategy $\sigma : A \to B$ is **thunkable** if it commutes with θ

$$A \xrightarrow{\Leftrightarrow} B$$

Thunking in $\mathcal{G}$ (not natural)

$$A \longrightarrow \Uparrow A$$

$$\theta_A = \begin{array}{cc} a_1^- & \\ & \bullet^- \\ & a_1^+ \\ a_2^- & \\ & a_2^+ \\ & a_3^- \\ a_3^+ & \end{array}$$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D}.\mathcal{S}.$$

neg. arenas	all arenas
neg. strat	all strats

Proposition

A strategy $\sigma : A \to B$ is **thunkable** if it commutes with θ

$$A \overset{\Leftrightarrow}{\longrightarrow} B$$

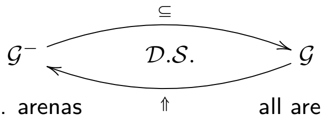
a^+

Thunking in $\mathcal{G}$ (not natural)

$$A \longrightarrow \Uparrow A$$

$$\theta_A = \begin{array}{cc} a_1^- & \\ & \bullet^- \\ & a_1^+ \\ a_2^- & \\ & a_2^+ \\ & a_3^- \\ a_3^+ & \end{array}$$

Duploids in non-alternating/concurrent games



neg. arenas | all arenas

neg. strat | all strats

Proposition

A strategy $\sigma : A \to B$ is **thunkable** if it commutes with θ

$$A \xrightarrow{\Leftrightarrow} B$$

$\dots$

$\dots$

a^+

Thunking in $\mathcal{G}$ (not natural)

$$A \longrightarrow \Uparrow A$$

$$\theta_A = \quad \begin{matrix} a_1^- & \\ & \bullet^- \\ & a_1^+ \\ a_2^- & \\ & a_2^+ \\ & a_3^- \\ a_3^+ & \end{matrix}$$

Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad \mathcal{D}.\mathcal{S}.$$

neg. arenas — all arenas

neg. strat — all strats

Proposition

A strategy $\sigma : A \to B$ is **thunkable** if it commutes with θ

$$A \overset{\Leftrightarrow}{\longrightarrow} B$$

$\ldots$

b^-

$\ldots$

a^+

Thunking in $\mathcal{G}$ (not natural)

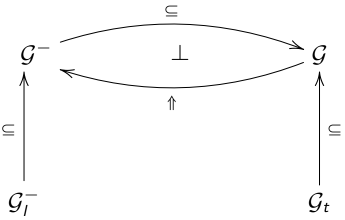
$$A \longrightarrow \Uparrow A$$

$$\theta_A = \quad \begin{array}{cc} a_1^- & \\ & \bullet^- \\ & a_1^+ \\ a_2^- & \\ & a_2^+ \\ & a_3^- \\ a_3^+ & \end{array}$$

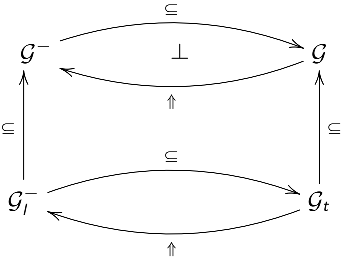
Duploids in non-alternating/concurrent games

$$\mathcal{G}^- \underset{\Uparrow}{\overset{\subseteq}{\rightleftarrows}} \mathcal{G} \qquad (\bot)$$

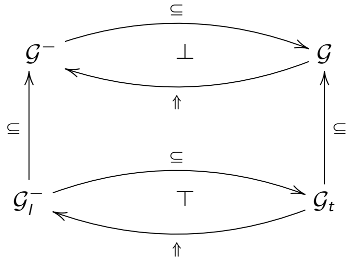
Duploids in non-alternating/concurrent games



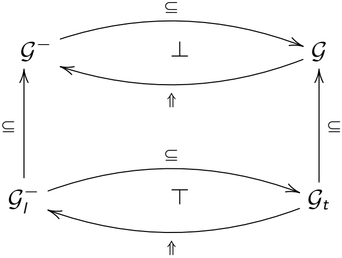
Duploids in non-alternating/concurrent games



Duploids in non-alternating/concurrent games

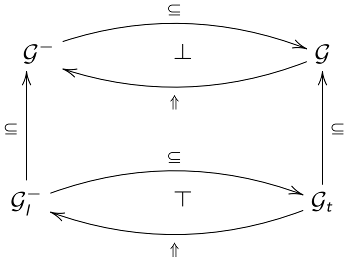


Duploids in non-alternating/concurrent games



$$
\begin{aligned}
|\mathcal{D}| &= |\mathcal{G}^-| + |\mathcal{G}| \\
\mathcal{D}(A,B) &= \mathcal{G}(A,B) \\
\mathcal{D}(N,M) &= \mathcal{G}(N,M) \\
\mathcal{D}(N,P) &= \mathcal{G}(N,P) &\cong\quad& \mathcal{G}^-(N, \Uparrow P) \\
\mathcal{D}(P,N) &= \mathcal{G}_t(P,N) &\cong\quad& \mathcal{G}_l^-(\Uparrow P, N)
\end{aligned}
$$

Duploids in non-alternating/concurrent games



$$
\begin{array}{rcl}
|\mathcal{D}| & = & |\mathcal{G}^-| + |\mathcal{G}| \\
\mathcal{D}(A, B) & = & \mathcal{G}(A, B) \\
\mathcal{D}(N, M) & = & \mathcal{G}(N, M) \\
\mathcal{D}(N, P) & = & \mathcal{G}(N, P) \quad \cong \quad \mathcal{G}^-(N, \Uparrow P) \\
\mathcal{D}(P, N) & = & \mathcal{G}_t(P, N) \quad \cong \quad \mathcal{G}_l^-(\Uparrow P, N)
\end{array}
$$

Duality/composition: **three duploids** for the price of one!