Polarities and classical constructiveness

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Overview

Proposition (Joyal)

Any Cartesian closed category $\mathscr{C}$ with an object 0 satisfying a natural isomorphism $0^{0^A} \simeq A$ is a boolean algebra.

Proof.

- The co-Cartesian structure is obtained by the duality 0^{-}.
- One has $\mathscr{C}(0 \times 0, 0) \simeq \mathscr{C}(0, 0^0)$ and $0^0 \simeq I$ is terminal thus one has $\pi_1 = \pi_2 : 0 \times 0 \to 0$.
- Thus for any $f, g : A \to 0$ one has $f = g$ because the two projections of $\langle f, g \rangle : A \to 0 \times 0$ are equal.
- Thus $\mathscr{C}(A \times 0^B, 0)$ contains at most one morphism, yet we have: $\mathscr{C}(A, B) \simeq \mathscr{C}(A, 0^{0^B}) \simeq \mathscr{C}(A \times 0^B, 0)$ ∎

Moral: Not easy to see which hypotheses we should relinquish.

Overview

Indirect interpretations

- Gödel-Gentzen $\neg\neg$-translation + Friedman-Dragalin's A-*translation*
  Π_2^0*-conservativity of Peano Arithmetic over Heyting Arithmetic*
- Gödel-Gentzen $\neg\neg$-translation + Gödel's Dialectica interpretation
  Interpretation of the axiom of dependent choice using bar recursion
- Translations into Girard's linear logic
  Denotational semantics satisfying $A = \neg\neg A$

Overview

Example

$$
\begin{aligned}
X(t_1, \ldots, t_n)^* &\stackrel{\text{def}}{=} X(t_1, \ldots, t_n) \\
(P \vee Q)^* &\stackrel{\text{def}}{=} P^* \vee Q^* \\
(P \wedge Q)^* &\stackrel{\text{def}}{=} P^* \wedge Q^* \\
(\exists x\, P)^* &\stackrel{\text{def}}{=} \exists x\, P^* \\
(P \to Q)^* &\stackrel{\text{def}}{=} \neg(P^* \wedge \neg Q^*) \\
(\forall x\, P)^* &\stackrel{\text{def}}{=} \neg \exists x\, \neg P^*
\end{aligned}
$$

Proposition

- *If $P \vdash Q$ classically then $P^* \vdash \neg\neg Q^*$ intuitionistically*
- *If $P \vdash Q$ classically then $P \vdash Q$ intuitionistically when P and Q are* purely positive *(transform an intuitionistic derivation of $P \vdash \neg\neg Q$ into one of $P \vdash (Q \to Q) \to Q$).*

Overview

Direct interpretations

- Gentzen's sequent calculus *(consistency of arithmetic)*, notably refined by Girard and Danos, Joinet and Schellinx
- Games (Gentzen; Novikoff; Coquand)
- Formulae-as-types, λ calculi with control operators: Griffin ($\lambda\mathcal{C}$); Parigot ($\lambda\mu$); Curien and Herbelin ($\bar{\lambda}\mu\tilde{\mu}$)
- Categorical interpretations of double-negation translations: Selinger; Hofmann and Streicher.
- Avigad's classical realisability
- Krivine's classical realisability
- Aschieri, Berardi and de' Liguoro's interactive realisability

and more

Direct interpretations

The $\lambda\mathcal{C}$ calculus

- Griffin showed that the control operator $\mathcal{C}$ has type $\neg\neg P \to P$
- The most convenient way of reducing terms is with abstract machines
- The call-by-name machine of Reus and Streicher:

$$
\begin{array}{rcl}
\langle t\,u \parallel \pi \rangle & \succ_n & \langle t \parallel u\cdot\pi \rangle \\
\langle \lambda x.t \parallel u\cdot\pi \rangle & \succ_n & \langle t[u/x] \parallel \pi \rangle \\
\langle \mathcal{C} \parallel t\cdot\pi \rangle & \succ_n & \langle t \parallel \mathsf{k}_\pi\cdot\mathsf{stop} \rangle \\
\langle \mathsf{k}_\pi \parallel t\cdot\pi' \rangle & \succ_n & \langle t \parallel \pi \rangle
\end{array}
$$

Direct interpretations

The meaning of Π^0_2-conservativity

- In the presence of side effects such as control, programs of certain types (functions, etc.) are opaque at runtime.
- But programs of type $P \vdash Q$ are still algorithms when P and Q are purely positive.

In other words, we do not assume that the behaviour of proofs has to be referentially transparent. Thus a proof of $A \vee \neg A$ needs not provide a decision procedure for A.

Direct interpretations

Lots of relationships

- Continuation-passing-style (CPS) translations that implement control operators are ¬¬-translations *(Murthy)* in a certain relationship with Gödel-Gentzen ¬¬-translations *(Lafont, Reus and Streicher and Laurent)*
- Girard's classical sequent calculus = refined ¬¬-translation + A-translation *(Murthy)*
- Avigad's classical realisability = ¬¬-translation + A-translation + modified realisability *(Avigad)*
- Interactive realisability = A-translation + modified realisability *(Aschieri and Berardi)*
- Krivine's classical realisability = ¬¬-translation + A-translation + modified realisability (+ Cohen's Forcing) *(Oliva and Streicher)*
- (2014) Formulae-as-types for Gödel's Dialectica interpretation *(Pédrot)*

Direct interpretations

Lots of relationships

- Coherent picture emerges
- Understanding the translations is at least as important as understanding the intuitionistic target
- Direct interpretations amount to studying both translations and their target at the same time

Expressive vs. fine-grained interpretations

- Expressiveness
  ex. *"Is it possible to realise the formula A?"*
  - Cut-elimination
  - Witness extraction
  - Consistency

  (easier)

- Understanding the fine details
  ex. *"Is there a behaviour common to all realisers of A?"*
  - In particular: type isomorphisms
    (thus: Equational theory with η laws)
  - Rewriting theory
  - Böhm theorem
  - Is there a canonical interpretation for classical logic?

Expressive vs. fine-grained interpretations

Here: Classical natural deduction that satisfies:

$$A \simeq \neg\neg A \quad , \quad \neg\forall x(A \to B) \simeq \exists x(A \wedge \neg B) \ \ldots$$

(*i.e.* reasoning by contrapositive)
with a clear constructive (*i.e.* programming) content:
the $\lambda\ell$ calculus, where ℓ is a control operator that we introduce

Guillaume Munch-Maccagnoni. Formulae-as-types for an involutive negation. In *Proceedings of the joint meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (CSL-LICS)*, 2014. To appear

Expressive vs. fine-grained interpretations

The $\lambda\mathcal{C}$ calculus is not fine-grained enough

- Cartesian closed (i.e. call-by-name λ calculus)
- A is a retract of $\neg\neg A$
- Example:

$$(\neg\forall \mathrm{x} \in \mathbb{N}\ A) \to \exists \mathrm{y} \in \mathbb{N}\ \neg A$$

  has a proof with the following skeleton:

$$\lambda xy.(\mathcal{C}\,\lambda k.(x\,\lambda e.(\mathcal{C}\,\lambda l.(k\,(y\,e\,l)))))$$

- Reasoning by contrapositive is non-trivial and counter-intuitive
  (Yet *e.g.* Krivine realises the axiom of dependent choice via its contrapositive)

Expressive vs. fine-grained interpretations

The $\lambda\mathcal{C}$ calculus is not fine-grained enough

Realising $(\neg\forall x \in \mathbb{N}\, A) \to \exists y \in \mathbb{N}\, \neg A$ should be as simple as:

1. Evaluating the argument until a stack of the form $n \cdot \pi$ appears
2. Return the pair (n, k_π) where k_π is the continuation of type $\neg A$

This is more or less what happens in the $\lambda\ell$ calculus

Formulae-as-types for an involutive negation

Polarisation

- Give a formal status to the polarities of connectives
  Goal: reconcile β-reductions with η-expansions
- For negative connectives, η-expansion delays evaluation. E.g. for $\rightarrow$:

$$tu \qquad vs \qquad \lambda x.tux$$

  Consequently, terms of a negative type are called by name
- For positive connectives, η-expansion forces evaluation. E.g. for $\vee$:

$$E[u] \qquad vs \qquad \mathsf{match}\ u\ \mathsf{with}\ (l(x).E[l(x)] \mid r(x).E[r(x)])$$

  Consequently, terms of a positive type are called by value

Formulae-as-types for an involutive negation

Polarisation

- Introduced by Girard in order to give a meaning to $A = \neg\neg A$ in classical sequent calculus (the logic LC)
- In LC, negation is defined by duality and is therefore not given as a connective
- Negation inverts the polarity
- The main insight of LC is, to me, the idea that the introduction rules of negation, taken as a connective, hide cuts

Formulae-as-types for an involutive negation

Polarisation

$$\dfrac{\dfrac{\Gamma, N \overset{\pi}{\vdash} \Delta}{\Gamma \vdash \neg N, \Delta} \qquad \Gamma', \neg N \overset{\pi'}{\vdash} \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \quad \triangleright \quad \dfrac{\dfrac{\dfrac{\dfrac{}{N \vdash N}}{\vdash \neg N, N} \qquad \Gamma', \neg N \overset{\pi'}{\vdash} \Delta'}{\Gamma' \vdash N, \Delta'} \qquad \Gamma, N \overset{\pi}{\vdash} \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

$$\dfrac{\Gamma' \overset{\pi'}{\vdash} \neg P, \Delta' \qquad \dfrac{\Gamma \overset{\pi}{\vdash} P, \Delta}{\Gamma, \neg P \vdash \Delta}}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \quad \triangleright \quad \dfrac{\Gamma \overset{\pi}{\vdash} P, \Delta \qquad \dfrac{\Gamma' \overset{\pi'}{\vdash} \neg P, \Delta' \qquad \dfrac{\dfrac{}{P \vdash P}}{P, \neg P \vdash}}{\Gamma', P \vdash \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Formulae-as-types for an involutive negation

Captured contexts are not continuations

- We show that Girard's logic is related to the idea in programming of having high-level access to the components of the contexts captured by control operators
- The type of captured contexts is therefore different from the type of continuations. Continuations are functions, and the contents of functions cannot be accessed in an immediate way
- It is obvious in "real-world" programming languages such as C that captured contexts are more primitive than continuations

Formulae-as-types for an involutive negation

Captured contexts are not continuations

One more motivation:

- Krivine simplifies reasoning in the λC calculus, by allowing certain *pseudo-types* in the left-hand side of implications.
- For technical reasons, an essential pseudo-type in Krivine's work is the set $\{\mathsf{k}_\pi \mid \pi \in X\}$. This also amounts to distinguishing a positive type of captured stacks from the type of continuations $X \to \bot$.
- The difference is, we will do so in a direct manner, making such types first class, in the sense that we define their meaning also when they are on the right-hand side of implications.

The $\lambda\ell$ calculus

- We introduce the positive type $\sim A$ of *inspectable stacks*, which is distinct from the negative type $A \to \bot$ of continuations
- We define negation in function of the polarity with:

$$\neg P \stackrel{\text{def}}{=} P \to \bot \qquad , \qquad \neg N \stackrel{\text{def}}{=} \sim N$$

  (defining negation in function of the polarity is reminiscent of Danos, Joinet and Schellinx)
- In the $\lambda\ell$ calculus we have the following isomorphisms:

$$\begin{aligned} P &\cong \sim(P \to \bot) \\ N &\cong (\sim N) \to \bot \\ \sim\forall x(A \to B) &\cong \exists x(A \land \sim B) \end{aligned}$$

The $\lambda\ell$ calculus

- The values that inhabit the type $\sim A$ are of the form $[\pi]$ where π is a context of the abstract machine
- We introduce combinators that let us access the contents of these inspectable stacks

$$D_{\to} : (\sim(A \to B)) \to (A \wedge \sim B)$$

$$D_{\forall} : (\sim \forall x\, N) \to \exists x \sim N$$

The $\lambda\ell$ calculus

Example

We derive $D_{\forall\rightarrow} : (\sim\forall\mathrm{x}(A \rightarrow B)) \rightarrow \exists\mathrm{x}(A \wedge \sim B)$ as follows:

$$D_{\forall\rightarrow} \stackrel{\text{def}}{=} \lambda x^{+}.\mathsf{let}\ y^{+}\ \mathsf{be}\ D_{\forall}\ x^{+}\ \mathsf{in}\ D_{\rightarrow}\ y^{+}$$

$D_{\forall\rightarrow}$ reduces as follows:

$$\langle D_{\forall\rightarrow} \parallel [V \cdot \pi] \cdot \pi_{+} \rangle \succ_{p}^{*} \langle (V, [\pi]) \parallel \pi_{+} \rangle$$

In pattern-matching notation, $D_{\forall\rightarrow}$ is the function:

$$\lambda[x \cdot \alpha].(x, [\alpha])$$

(compare to the term of the λC calculus)

The $\lambda\ell$ calculus

A captured stack $[\pi]$ can be re-installed as the context of another term t by the constant send[1]:

$$\langle \mathsf{send} \parallel [\pi]\cdot t\cdot \pi' \rangle \succ_p \langle t \parallel \pi \rangle$$

In other words, the constant send converts a captured stack into a continuation:

$$\mathsf{send} : (\sim A) \to A \to \bot$$

[1] For didactic reasons, the present versions of send and ℓ (next slide) are undelimited variants of the operators from the article.

The $\lambda\ell$ calculus

The operator responsible for the apparition of inspectable stacks is ℓ:

$$\ell : (A \to \bot) \to {\sim}A$$

This operation is formally described by introducing the j_π operator (analogous to the k_π of $\lambda\mathcal{C}$).
The operator ℓ saves with j the context π in which ℓ is applied:

$$\langle \ell \parallel t \cdot \pi \rangle \succ_p \langle t \parallel \mathsf{j}_\pi \cdot \mathsf{stop} \rangle$$

Once the operator j_π comes in head position, it captures the stack and restores the context π:

$$\langle \mathsf{j}_\pi \parallel \pi' \rangle \succ_p \langle [\pi'] \parallel \pi \rangle$$

The $\lambda\ell$ calculus

Contributions in details

- A language of untyped realisers (quasi-proofs)
- The issue of $\bot$ in an untyped setting is solved with control delimiters (inspired by Ariola, Herbelin and Sabry; Herbelin and Ghilezan)
- $\lambda\ell$ is provided with an equational theory by embedding into a sequent calculus whose cut-elimination is confluent ($\mathsf{L}_{\mathrm{pol},\hat{\mathrm{tp}}}$ inspired by Curien and Herbelin's $\bar{\lambda}\mu\tilde{\mu}$)
- Double-negation translations for $\lambda\ell$ and $\mathsf{L}_{\mathrm{pol},\hat{\mathrm{tp}}}$ simulate reductions and preserve equivalences (hence strong normalisation of typed terms and coherence)
- A direct computational interpretation of polarities, which can be adapted for non-classical Call-by-Push-Value models
- Contains De Groote-Saurin's $\Lambda\mu$ and variants of the the $\mathrm{shift}_0/\mathrm{reset}_0$ operators

The $\lambda\ell$ calculus

Contributions in details

The catch is: we give up associativity of composition when the middle map is from positive to negative (*duploids*, see the second part)

Thank you