Note on models of polarised intuitionistic logic

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Following renewed interest in duploids arising from the exponential comonad of linear logic (the construction describing polarised intuitionistic translations into linear logic), I summarise here various remarks:

- about a decomposition of Girard's "boring" translation as the expression of call-by-value in call-by-name, dual to how thunks is used to express call-by-name in call-by-value (object of a talk given in June 2012 at LDP, Marseille, entitled *"Runnable monads in models of the λ-calculus"*),
- about the coincidence between linear CPS translations and Girard's translations of intuitionistic logic into linear logic (object of a remark in my PhD thesis and of a private communication from Streicher in October 2013),
- about a completeness property of historical models of linear logic in the above context (object of an untitled note circulated privately around July 2014),
- about a rational reconstruction of these translations with the Linear Call-by-Push-Value (part of a talk given in March 2016 at IRIF, Paris, entitled *"Adjunction models and polarised calculi for modelling effects and resources"*).

1 Runnable monads in models of the λ-calculus

Historical models of linear logic satisfy an equalizing property not accounted for by traditional axiomatic accounts of linear logic models (symmetric monoidal adjunctions, see Melliès, 2009). The following property holds for the relational model (including generalisations with weights), coherence spaces (both with set and multiset exponentials), and hypercoherences:

Proposition 1. *Let us write $\mathscr{V}$ the linear category, $! : \mathscr{V} \to \mathscr{V}$ the linear exponential comonad, and $\varepsilon : ! \to 1$ the co-unit. Then ε is a coequalizer of the pair $\varepsilon_!, !\varepsilon : !! \to !$.*

As a consequence, Führmann's completeness property (1999) applies (in the dual) as described next. This property is best understood via the Curry-Howard correspondence by dualising Führmann's construction of thunks.

Every model of linear logic $\mathscr{V}$ gives rise to a model of intuitionistic logic in the Kleisli category $\mathscr{V}_!$; the latter is equipped with the dual of a thunk which we call a *runnable monad*. Thunks express call by name in call by value (Hatcliff and Danvy, 1997). Dually, the runnable monad expresses call by value in call by name, and therefore induces a translation of call by value into linear logic, in which context it coincides with Girard's "boring translation" (Munch-Maccagnoni, 2013). Such models of intuitionistic logic in which the order of evaluation matters are *polarised*.

The polarised structure is *depolarised* when one of the following equivalent properties are satisifed (Munch-Maccagnoni, 2014): the evaluation order does not matter (unrestricted β-reduction is satisfied), the runnable monad is isomorphic to the identity functor, and $\delta : ! \to !!$ is an isomorphism. This cannot happen, though, if the model of linear logic has an isomorphism $!o \multimap o \cong o$ so as to interpret untyped λ-calculus. Indeed, if the model is depolarised, then $\mathscr{V}_!$ is bi-cartesian closed, and therefore the impossibility results in Huwig and Poigné (1990) apply.

2 Führmann's completeness

A runnable monad determines a notion of linearity: Führmann's thunkability in the dual. Führmann's result then states that Proposition 1 is equivalent to the following:

Corollary 2. *The subcategory of $\mathscr{V}_!$ whose morphisms are linear is isomorphic to $\mathscr{V}$, in the category of categories equipped with a comonad and functors that preserve the co-monad structure.*

This justifies in particular the terminology "linear": it generalizes Girard's notion of linearity originating from the historical models of linear logic to more general models of computation. (Specializations of this theorem to a notion of isomorphisms that preserves the structure of connectives of linear logic have since been studied by Blute, Cockett, and Seely, 2015.)

The phenomenon is similar to the relationship, among models of classical logic, between response categories and control categories (Selinger, 2001). Any response category (indirect model) gives rise to a control category (direct model), and any control category arises from a response category (the category of linear morphisms) in this way. This relationship between indirect and direct models is a reflection (Führmann, 1999).

This result has sometimes been used to support the point of view that control categories are "the same" as response categories. However, the response category arising from a control category is not necessarily isomorphic to the one that gave rise to the control category (Selinger, 2001). But, the additional property given by Proposition 1 characterises the reflective full subcategory of indirect models equivalent to the category of direct models. In other words, in the case of historical models of linear logic giving rise to polarised intuitionistic logic models, one indeed has an isomorphism between the indirect model and the direct model (which the syntaxes for linear logic do not necessarily reflect).

2.1 Example in Rel

Let $\mathcal{R}el$ be the category of sets and relations. Following Ehrhard (2012) we write multi-sets $m = [a_1, \dots, a_n] \in \mathcal{M}_{\text{fin}}$ and their union additively. The finite multi-set co-monad is written $(!, \varepsilon, \delta)$. λ-terms are interpreted in the Kleisli category $\mathcal{R}\boldsymbol{el}_!$, that is, they are represented by sets f of relations $(m, b) \in f$ with $m \in \mathcal{M}_{\text{fin}}(A)$ and $b \in B$ for some sets A, B.

Proposition 3. *$\mathcal{R}\boldsymbol{el}_!$ has a runnable monad (T, ρ, η) defined with:*

- $TA \stackrel{\text{def}}{=} \mathcal{M}_{fin}(A)$;
- $Tf \stackrel{\text{def}}{=} \left\{([m_1 + \cdots + m_n], [a_1, \ldots, a_n]) \;\middle|\; n \in \mathbb{N} \text{ and } \forall i,\ (m_i, a_i) \in f\right\}$;
- $\rho_A \stackrel{\text{def}}{=} \left\{([[a]], a) \;\middle|\; a \in A\right\}$;
- $\eta_A \stackrel{\text{def}}{=} \{(m, m) \mid m \in \mathcal{M}_{fin}(A)\}$.

Proof. This is an instance of Führmann's construction (1999), seen in the dual. ∎

Definition 4. A morphism $f \in \mathcal{R}\boldsymbol{el}_!(A, B)$ is *linear* if $\rho_B \circ Tf = f \circ \rho_A$ (or equivalently $f \circ !\varepsilon_A = f \circ \varepsilon_{!A}$ in $\mathcal{R}\boldsymbol{el}$).

Proposition 5. *A morphism $f \in \mathcal{R}\boldsymbol{el}_!(A, B)$ is linear if and only if $(m, b) \in f \implies \#m = 1$. In particular ε is a coequalizer of the pair $\varepsilon_!, !\varepsilon : !! \to !$.*

Proof. For any $f \in \mathcal{R}\boldsymbol{el}_!(A, B)$, the morphisms $f^* = \rho_B \circ Tf$ and $f \circ \rho_A$ in $\mathcal{R}\boldsymbol{el}_!(TA, B)$ are defined with:

$$
\begin{aligned}
f^* &= \left\{([m], a) \;\middle|\; (m, a) \in f\right\} \\
f \circ \rho_B &= \left\{([[a_1], \ldots, [a_n]], b) \;\middle|\; ([a_1, \ldots, a_n], b) \in f\right\}
\end{aligned}
$$

Thus $f^* = f \circ \rho_B$ if and only if $(m, b) \in f \Rightarrow \#m = 1$. This shows that for any $f \in \mathcal{R}\boldsymbol{el}(!A, B)$ such that $f \circ \varepsilon_{!A} = f \circ !\varepsilon_A$, there exists $g \in \mathcal{R}\boldsymbol{el}(A, B)$ such that $f = g \circ \varepsilon_A$, determined uniquely by

$$
(a, b) \in g \iff ([a], b) \in f\ .
$$

∎

3 The polarised case

The generalisation of Führmann's result with polarities is provided with duploids arising from adjunctions (Munch-Maccagnoni, 2014). One motivation of duploids is to relax the self-duality of dialogue categories (Melliès, 2009) so as to account for settings in which negation is not involutive (e.g. Munch-Maccagnoni, 2013, Section II.2.6). If one is interested in seeing models of intuitionistic logic arise from ones of linear logic:

1. one can start from the decomposition of response categories into dialogue categories with resource adjunctions (Melliès, 2009; Melliès and Tabareau, 2010), but now composing with the resource adjunction on one side only:

   $$
   \mathscr{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathscr{V} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathscr{V}^{\text{op}}
   $$

   (Data: $\mathscr{V}$ a distributive dialogue category, $\mathscr{M}$ a cartesian category, $\mathscr{V} \rightleftarrows \mathscr{V}^{\text{op}}$ the adjunction of negation with itself, $\mathscr{M} \rightleftarrows \mathscr{V}$ a symmetric monoidal adjunction.)

2. alternatively, if $\mathscr{V}$ above is closed, as is the case for models of linear logic, one can also consider the duploid arising from the resource adjunction $\mathscr{M} \underset{\perp}{\rightleftarrows} \mathscr{V}$ alone. (As is well-known in this case the monad on $\mathscr{M}$ is commutative.)

The duploid construction subsumes call-by-value and call-by-name translations. The variety associated to the adjunctions above describe:

1. for the adjunction $\mathscr{M} \underset{\perp}{\rightleftarrows} \mathscr{V}^{\mathrm{op}}$: CPS translations of *"non-linear languages with linear control"* summarised in Hasegawa (2004) (encompassing notably linearly-used continuations by Berdine, O'Hearn, Reddy, and Thielecke, 2000),

2. for the adjunction $\mathscr{M} \underset{\perp}{\rightleftarrows} \mathscr{V}$: polarised translations of intuitionistic logic described by Girard (1993, 2007); Liang and Miller (2007, 2009),

as we will see in more detail in the next section.

Remark 6. In fact, when $\mathscr{V}$ is $*$-autonomous, as in the case of models of linear logic, the duploids arising from the intuitionistic polarised translation and from the linear CPS are in an obvious equivalence, because the adjunctions we started with differ up to composition by an adjunction $\mathscr{V} \underset{\perp}{\rightleftarrows} \mathscr{V}^{\mathrm{op}}$ which happens to be an equivalence.

I believe that this remark captures the essence of a note by Streicher (2013) suggesting the coincidence between Girard's translations and linear CPS (in call by name and by value), communicated privately after Munch-Maccagnoni, 2013, Chapter III based on this coincidence an analysis of delimited CPS translations as polarised translations into linear logic.

Equivalence between duploids is in the sense of the 2-category of duploids, duploid functors, and natural transformations whose components are linear and thunkable, which provides the appropriate notion. But in the case where negation is strictly involutive, as is usually the case in syntaxes of linear logic, then the equivalence is an isomorphism, and one already has an isomorphism between the adjunctions $\mathscr{M} \underset{\perp}{\rightleftarrows} \mathscr{V}$ and $\mathscr{M} \underset{\perp}{\rightleftarrows} \mathscr{V}^{\mathrm{op}}$.

3.1 Completeness

The case of the runnable monad mentioned previously corresponds to taking $\mathscr{M} = \mathscr{V}_!$. Another natural choice, considered in Ehrhard (2016) is $\mathscr{M} = \mathscr{V}^!$ the category of coalgebras. Either way, it is a general fact that the monad arising on $\mathscr{V}_!$ and $\mathscr{V}^!$ satisfies the equalizing property dual to Proposition 1 (see e.g. Barr and Wells, 1985, Proposition 3.3.7 in the dual). This property and its dual form the criterion to generalise Führmann's completeness to duploids. Thus:

Corollary 7. *In the case of historical models of linear logic, the indirect polarised model $\mathscr{M} \underset{\perp}{\rightleftarrows} \mathscr{V}$ is isomorphic, as an adjunction, to the one between the categories of thunkable and linear morphisms in its associated duploid.*

We refer the reader to Munch-Maccagnoni (2014) for the details of notions and constructions. This motivates, I believe, the study of linear logic via their associated Call-by-Push-Value model.

4 Models of Call-by-Push-Value arising from linear logic

The notion of Call-by-Push-Value model (Levy, 2005) coincides with that of (indirect) polarised intuitionistic logic $\mathbf{LJ}_p^\eta$ model (Curien, Fiore, and Munch-Maccagnoni, 2016), so that one can interpret polarised intuitionistic logic into any CBPV model, not just ones arising from linear logic. We now recall how the previous polarised intuitionistic situations give rise to CBPV models following general considerations. We use the notions and results from Curien, Fiore, and Munch-Maccagnoni, 2016 where more detail can be found.

- Any dialogue category (Melliès, 2009) with distributive structure $\mathcal{V}$ gives rise to a Linear CBPV model $\underline{\mathcal{V}} \underset{\perp}{\rightleftarrows} \underline{\mathcal{V}}^{\mathrm{op}}$.
- If in addition $\mathcal{V}$ is closed, in other words is a distributive SMCC, then it gives rise to a Linear CBPV model $\underline{\mathcal{V}} \underset{\perp}{\rightleftarrows} \underline{\mathcal{V}}$.
- Any symmetric monoidal adjunction $\mathcal{M} \underset{M}{\overset{L}{\rightleftarrows}} \mathcal{V}$, such as for $\mathcal{M} = \mathcal{V}_!$ for Seely categories and $\mathcal{M} = \mathcal{V}^!$ for linear categories—in which cases $\mathcal{M}$ is cartesian—(Bierman, 1995; Melliès, 2009), enriches into an adjunction $\underline{\mathcal{M}} \underset{\perp}{\rightleftarrows} L^\star\underline{\mathcal{V}}$ (where $L^\star$ denotes precomposition of presheaves with L).
- Any $\mathbf{ILL}_p^\eta$ model $(\mathcal{M} \underset{\perp}{\rightleftarrows} \mathcal{V}, \underline{\mathcal{V}} \underset{\perp}{\rightleftarrows} \underline{\mathcal{S}})$ (a symmetric monoidal adjunction over a Linear CBPV adjunction where $\mathcal{M}$ is cartesian) gives rise to a CBPV model $\underline{\mathcal{M}} \underset{\perp}{\rightleftarrows} L^\star\underline{\mathcal{S}}$ by composing the image of the LCBPV adjunction by $L^\star$ with the previous one.

Therefore we have recipes for CBPV models:

1. for any distributive dialogue category with a cartesian resource modality, as the adjunction

   $$\underline{\mathcal{M}} \underset{\perp}{\rightleftarrows} L^\star\underline{\mathcal{V}^{\mathrm{op}}}$$

   given by the bijections

   $$\mathcal{M}(\Gamma \times P, M(Q_0 \multimap \perp)) \cong \mathcal{V}(L\Gamma \otimes Q_0, LP \multimap \perp)$$

   natural in $\Gamma, P \in \mathcal{M}$ and in $Q_0 \in \mathcal{V}$,

2. for any SMCC with a cartesian resource modality, as the adjunction

   $$\underline{\mathcal{M}} \underset{\perp}{\rightleftarrows} L^\star\underline{\mathcal{V}}$$

   given by the bijections

   $$\mathcal{M}(\Gamma \times P, MN) \cong \mathcal{V}(L\Gamma \otimes LP, N)$$

   natural in $\Gamma, P \in \mathcal{M}$ and in $N \in \mathcal{V}$, which therefore enrich the adjunctions in Section 3 using the fact that the left adjoint L is strong monoidal. As previously, they differ by the composition of an adjunction $L^\star\underline{\mathscr{V}} \underset{\leftarrow}{\overset{\rightarrow}{\bot}} L^\star\underline{\mathscr{V}}^{\mathrm{op}}$ which, when the closed structure of $\mathscr{V}$ is $*$-autonomous, is an equivalence.

More specifically, one has:

1. CBPV models

$$\underline{\mathscr{V}_!} \underset{\overset{\longleftarrow}{\downarrow}}{\overset{\overset{\uparrow}{\longrightarrow}}{\bot}} F_!^\star\underline{\mathscr{V}^{\mathrm{op}}}$$

where:

$$\downarrow N = N^{\mathrm{op}} \multimap \bot \qquad\qquad \uparrow P = (!P \multimap \bot)^{\mathrm{op}}$$

and where powers are given with:

$$P \twoheadrightarrow N = (!P \otimes N^{\mathrm{op}})^{\mathrm{op}}$$

Call-by-value and call-by-name sequents and arrows are therefore found as follows:

$$\begin{aligned}
\mathscr{P}(\Gamma_+ \otimes P, Q) &\cong \underline{\mathscr{V}^{\mathrm{op}}}_{!\Gamma}(\uparrow P, \uparrow Q) = \mathscr{V}(!\Gamma \otimes (!Q \multimap \bot), (!P \multimap \bot))\\
P \to_{\mathrm{CBV}} Q &= \downarrow(P \twoheadrightarrow \uparrow Q) = (!P \otimes (!Q \multimap \bot)) \multimap \bot\\
\mathscr{N}(\Gamma_\ominus \,\&\, N, M) &\cong \underline{\mathscr{V}^{\mathrm{op}}}_{!\downarrow\Gamma}(\uparrow\downarrow N, M) = \mathscr{V}(!(\Gamma^{\mathrm{op}} \multimap \bot) \otimes M^{\mathrm{op}}, !(N^{\mathrm{op}} \multimap \bot) \multimap \bot)\\
N \to_{\mathrm{CBN}} M &= \downarrow N \twoheadrightarrow M = (!(N^{\mathrm{op}} \multimap \bot) \otimes M^{\mathrm{op}})^{\mathrm{op}}
\end{aligned}$$

where we have written $\cdot^{\mathrm{op}}$ the formal dual for legibility.

2. CBPV models

$$\underline{\mathscr{V}^!} \underset{\overset{\longleftarrow}{\downarrow}}{\overset{\overset{\uparrow}{\longrightarrow}}{\bot}} U^\star\underline{\mathscr{V}}$$

where

$$\downarrow N = !N \qquad\qquad \uparrow P = P$$

and where powers are given with:

$$P \twoheadrightarrow N = P \multimap N$$

Call-by-value and call-by-name sequents and arrows are therefore found as follows:

$$\begin{aligned}
\mathscr{P}(\Gamma_+ \otimes P, Q) &\cong \underline{\mathscr{V}}_{\Gamma}(\uparrow P, \uparrow Q) = \mathscr{V}(\Gamma \otimes P, Q)\\
P \to_{\mathrm{CBV}} Q &= \downarrow(P \twoheadrightarrow \uparrow Q) = !(P \multimap Q)\\
\mathscr{N}(\Gamma_\ominus \,\&\, N, M) &\cong \underline{\mathscr{V}}_{\downarrow\Gamma}(\uparrow\downarrow N, M) = \mathscr{V}(!\Gamma \otimes !N, M)\\
N \to_{\mathrm{CBN}} M &= \downarrow N \twoheadrightarrow M = !N \multimap M
\end{aligned}$$

3. CBPV models

$$\underline{\mathscr{V}_!} \underset{\downarrow}{\overset{\uparrow}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} F_!^\star \underline{\mathscr{V}}$$

where

$$\downarrow N = N \qquad\qquad \uparrow P = \,!P$$

and where powers are given with:

$$P \twoheadrightarrow N = \,!P \multimap N$$

Call-by-value and call-by-name sequents and arrows are therefore found as follows:

$$\begin{aligned}
\mathscr{P}(\Gamma_+ \otimes P, Q) &\cong \underline{\mathscr{V}}_{!\Gamma}(\uparrow P, \uparrow Q) = \mathscr{V}(!\Gamma \otimes !P, !Q)\\
P \to_{\mathrm{CBV}} Q &= \downarrow(P \twoheadrightarrow \uparrow Q) = \,!P \multimap !Q\\
\mathscr{N}(\Gamma_\ominus \,\&\, N, M) &\cong \underline{\mathscr{V}}_{!\downarrow\Gamma}(\uparrow\downarrow N, M) = \mathscr{V}(!\Gamma \otimes !N, M)\\
N \to_{\mathrm{CBN}} M &= \downarrow N \twoheadrightarrow M = \,!N \multimap M
\end{aligned}$$

As one can see, one finds back CPS of non-linear languages with linear control (Hasegawa, 2004), and two formulations of the translations of call-by-value and call-by-name intuitionistic logic into linear logic (Girard, 1987). Generalising the consideration of the Kleisli categories $\mathscr{P}$ and $\mathscr{N}$ to that of the duploid arising from the adjunction accounts for the more general polarised translation in Girard (2007, 12.B.1).

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