

A Theory of Effects and Resources: Adjunction Models and Polarised Calculi

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Joint work with



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Curry-Howard-Lambek correspondence

for

effectful computation

(monads, call-by-push-value)

and resource-aware computation

(linear logic, resource modalities)

Outline

Introduction

Curry-Howard-Lambek

Computational Effects and Linear Logic

“Extending Curry-Howard-Lambek”?

Adjunction models

Presheaf enrichments

Functions, sums, resource modalities

Polarisation

A paradigm of evaluation order

Polarised calculi

Depolarisation

Summary

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Summary



Curry-Howard-Lambek

- A correspondence between programming languages, proof systems and algebraic structures
- e.g. *types / formulae / objects*
 $A, B ::= 1 \mid A \times B \mid A \rightarrow B \mid A + B$
- foundations

Curry-Howard-Lambek

Phenomenon *Similar structures appearing in programming, logic, and algebra*

Explanation *A paradigm of higher-order computation*

Role of the paradigm

- What is the starting point? What do we agree upon?
- Let us formulate and share scientific questions
- Criteria of scientific validity
- Go further:
focus, study in more details, challenge the paradigm

Computational Effects and Linear Logic

Similar structures appearing in programming and logic

Computational effects

- Moggi: decompose effectful computation as $A \rightarrow TB$
- T is a *monad* that encapsulates effects
- Levy: refine into *adjunction models* (call-by-push-value: CBPV)

Linear logic

- Girard: decompose intuitionistic implication as $!A \multimap B$
- $!$ is a *co-monad* that manages resources (e.g. complexity)
- Seely, Benton and others: refine into a *symmetric monoidal adjunction*

Computational Effects and Linear Logic

Similar structures appearing in programming and logic

- Monads and co-monads decomposed into adjunctions
- Both call-by-value and call-by-name computations each time
- Tell-tale sign in formalisms: presence of *stacks* (or applicative contexts, abstract machines, *defunctionalised continuations*, left-introduction rules...)

“Extending Curry-Howard-Lambek”?

- ~~Proofs are programs~~

Instead:

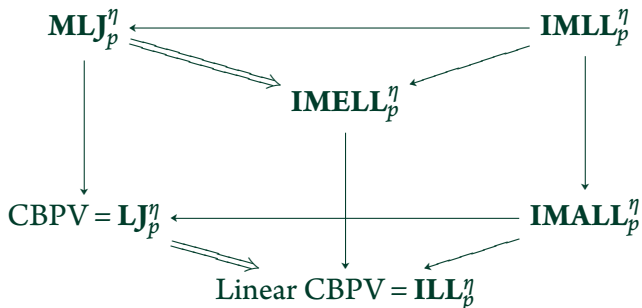
- Looking for paradigms to guide us
- Normal scientific process

In this article:

- Decomposition of CBPV into Linear CBPV, that distinguishes the effect adjunction from the resource adjunction.
- Interpretation as models for calculi of intuitionistic logic (**LJ**) and intuitionistic linear logic (**ILL**) via polarisation: a paradigm of evaluation order.

“Extending Curry-Howard-Lambek”?

Incremental presentation:



LJ: intuitionistic logic

ILL: intuitionistic linear logic

η : “with evaluation order”

M: multiplicatives ($\otimes, \multimap, 1$)

A: additives ($\oplus, \&, 0, \top$)

E: exponential (!)

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Presheaf enrichments

Cartesian enrichment

- \mathcal{V} cartesian category of *values*. ($P, Q, \Gamma \dots \in \mathcal{V}$)
- Category $\underline{\mathcal{C}}$ enriched on \mathcal{V} -presheaves: morphisms $A \rightarrow B$ under context $\Gamma \in \mathcal{V}$

$$\underline{\mathcal{C}}_{\Gamma}(A, B)$$

- Identity and composition under a context:

$$\text{id}_X^{(\Gamma)} \in \underline{\mathcal{C}}_{\Gamma}(X, X) \quad \circ_{X,Y,Z}^{(\Gamma)} : \underline{\mathcal{C}}_{\Gamma}(Y, Z) \times \underline{\mathcal{C}}_{\Gamma}(X, Y) \rightarrow \underline{\mathcal{C}}_{\Gamma}(X, Z)$$

- Ex: $\underline{\mathcal{V}}$: $\underline{\mathcal{V}}_{\Gamma}(P, Q) = \mathcal{V}(\Gamma \times P, Q)$
- Category of *stacks* $\underline{\mathcal{S}}$ ($N, M \dots \in \underline{\mathcal{S}}$)
given with an enriched adjunction $F \dashv G$:

$$\underline{\mathcal{S}}_{\Gamma}(FP, N) \cong \underline{\mathcal{V}}_{\Gamma}(P, GN)$$

Presheaf enrichments

Symmetric monoidal enrichment

- \mathcal{L} symmetric monoidal category of *linear values*. ($P, Q, \Gamma \dots \in \mathcal{L}$)
- Category $\underline{\mathcal{C}}$ enriched on \mathcal{L} -presheaves: morphisms $A \rightarrow B$ under context $\Gamma \in \mathcal{L}$
(Day's convolution monoidal structure on presheaves)
- Identity and composition under a linear context:

$$\text{id}_X \in \underline{\mathcal{C}}_I(X, X) \quad \circ_{X,Y,Z}^{(\Gamma, \Gamma')} : \underline{\mathcal{C}}_{\Gamma'}(Y, Z) \times \underline{\mathcal{C}}_{\Gamma}(X, Y) \rightarrow \underline{\mathcal{C}}_{\Gamma \otimes \Gamma'}(X, Z)$$

- Ex: $\underline{\mathcal{L}}$: $\underline{\mathcal{L}}_{\Gamma}(P, Q) = \mathcal{L}(\Gamma \otimes P, Q)$
- Category of *stacks* $\underline{\mathcal{S}}$ ($N, M \dots \in \underline{\mathcal{S}}$)
given with an enriched adjunction $F \dashv G$:

$$\underline{\mathcal{S}}_{\Gamma}(FP, N) \cong \underline{\mathcal{L}}_{\Gamma}(P, GN)$$

Functions, sums, resource modalities

Function space

- $\mathcal{V} / \mathcal{L}$ -powers:

$$\underline{\mathcal{S}}_{\Gamma}(M, P \rightarrow N) \cong \underline{\mathcal{S}}_{\Gamma \otimes P}(M, N)$$

- **MLJ**_p^η models: $\underline{\mathcal{V}} \xleftrightarrow{\perp} \underline{\mathcal{S}}$ where $\underline{\mathcal{S}}$ has \mathcal{V} -powers
 → is →
 Coincides with EC models (Egger, Møgelberg, Simpson)
- **IMLL**_p^η models: $\underline{\mathcal{L}} \xleftrightarrow{\perp} \underline{\mathcal{S}}$ where $\underline{\mathcal{S}}$ has \mathcal{L} -powers
 → is \multimap
- Optionally, cartesian product on $\underline{\mathcal{S}}$

Functions, sums, resource modalities

Sums

- $\mathcal{V} / \mathcal{L}$ (linearly) distributive ($\Gamma \otimes (P + Q) \cong \Gamma \otimes P + \Gamma \otimes Q$)
- Enrich on *distributive presheaves*:

$$\underline{\mathcal{C}}_0(X, Y) \cong 1 \quad \underline{\mathcal{C}}_{\Gamma+\Gamma'}(X, Y) \cong \underline{\mathcal{C}}_{\Gamma}(X, Y) \times \underline{\mathcal{C}}_{\Gamma'}(X, Y)$$

- **LJ_p^η** model: **MLJ_p^η** model +
ℳ distributive, $\underline{\mathcal{S}}$ cartesian with distributive hom-presheaves
Coincides with Call-by-push-value adjunction models (Levy)
- **IMALL_p^η** model: **IMLL_p^η** model +
ℳ lin. distributive, $\underline{\mathcal{S}}$ cartesian with distributive hom-presheaves

Functions, sums, resource modalities

Resource modalities

- **IMELL**_p^η model: **IMLL**_p^η model $\underline{\mathcal{L}} \xrightarrow{\perp} \underline{\mathcal{S}} +$
symmetric monoidal adjunction $\mathcal{V} \xrightarrow{\perp} \mathcal{L}$ (resource modality)
Examples: **IMELL** (\mathcal{L} SMCC), dialogue categories with
resource modalities (Melliès and Tabareau)
- Every resource modality $\mathcal{V} \xrightarrow{L\perp} \mathcal{L}$ enriches into an
adjunction

$$\underline{\mathcal{V}} \xrightarrow{\perp} \underline{\mathcal{L}}_L$$

($\underline{\mathcal{C}}_L$ defined by precomposing presheaves with L , i.e.

$$(\underline{\mathcal{C}}_L)_A = \underline{\mathcal{C}}_{LA}$$

- This yields an **MLJ**_p^η model by composing adjunctions

$$\underline{\mathcal{V}} \xrightarrow{\perp} \underline{\mathcal{L}}_L \xrightarrow{\perp} \underline{\mathcal{S}}_L$$

and defining $P \rightarrow N \stackrel{\text{def}}{=} LP \multimap N$ (“Girard” translation)

Functions, sums, resource modalities

Linear Call-by-push-value

- \mathbf{ILL}_p^η model: \mathbf{IMALL}_p^η model + resource modality

Ex: Melliès' dialogue categories/chiralities with a resource modality and co-products (“ \mathbf{LL}_p^η ” as \mathbf{ILL}_p^η + involutive negation); linear local store models (to investigate).

- \mathbf{ILL}_p^η models give rise to \mathbf{LJ}_p^η models by composing adjunctions as before

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A paradigm of evaluation order

let x^A be t in u

- Lazy: u before t
- Strict: t before u
- Types are positive or negative: $A, B ::= P \mid N$
- Positive types: $P, Q ::= 1 \mid X^+ \mid A \otimes B \mid A \oplus B \mid 0 \mid !A$
Evaluate strictly
(Polarities do not match focusing properties)
- Negative types: $N, M ::= \top \mid X^- \mid A \& B \mid A \rightarrow B$
Evaluate lazily

A paradigm of evaluation order

A thought experiment:

let y^A be t in let x^B be u in v $\stackrel{?}{=} \text{let } x^B \text{ be } (\text{let } y^A \text{ be } t \text{ in } u) \text{ in } v$

A	B	
lazy	lazy	=
strict	strict	=
lazy	strict	=
strict	lazy	\neq

A paradigm of evaluation order

ML Define lazy composition with **thunking**

$$\text{let } y = f \ x \ \text{in } h \ (\text{fun } () \ \rightarrow g \ y) \neq h \ (\text{fun } () \ \rightarrow g \ (f \ x))$$

Haskell Use \$! as strict composition

$$(\backslash y \rightarrow h \ (g \ y)) \ \$! \ (f \ x) \neq h \ (g \ \$! \ (f \ x))$$

A paradigm of evaluation order

FoSSaCS 2014 (M.)

- Axiomatization of a non-associative composition
- Reflection theorem with adjunctions
- Difference with focusing ($\Downarrow \dashv \Uparrow$ vs. $\Uparrow \dashv \Downarrow$)

Polarised calculi

LJ_p^η and ILL_p^η

- Abstract-machine-like calculi:
 - Computation as reduction of pairs $\langle \text{expressions}, \text{context} \rangle$
 - Constructs are defined as solutions to their abstract-machine transitions
 - Evaluation order determined by the type
- λ -calculi (call-by-name, call-by-value or both) obtained with syntactic sugar
- Type systems are the sequent calculi LJ and ILL , however cut-elimination follows an order
- “Barendregt-style”: Generation lemma, Decidability of typing, Subject reduction, Confluence, and similar properties of the λ -calculus in the style of Barendregt

Polarised calculi

Intepretation

- Interpret sequents $\Gamma, A \vdash B$ into the sets:

$$\underline{\mathcal{S}}_{\Gamma^+}(FA^+, B^-) \cong \underline{\mathcal{V}}_{\Gamma^+}(A^+, GB^-)$$

- $^+$ and $^-$ add F and G wherever necessary
- If A is positive, interpret cut as

$$\underline{\mathcal{S}}_{\Gamma_2^+}(FA, B^-) \times \underline{\mathcal{S}}_{\Gamma_1^+}(FI, FA) \rightarrow \underline{\mathcal{S}}_{\Gamma_1^+ \otimes \Gamma_2^+}(FI, B^-)$$

Generalises the Kleisli composition for the monad GF

- If A is negative, interpret cut as

$$\underline{\mathcal{V}}_{\Gamma_2^+}(GA, GB^-) \times \underline{\mathcal{V}}_{\Gamma_1^+}(I, GA) \rightarrow \underline{\mathcal{V}}_{\Gamma_1^+ \otimes \Gamma_2^+}(I, GB^-)$$

Generalises the Kleisli composition for the co-monad FG

Polarised calculi

Structural rules

- Do not represent structural rules explicitly
- All at once:

$$\frac{\Gamma \vdash t:A \mid}{\Gamma' \vdash t[\sigma]:A \mid} (\sigma \vdash)$$

σ : substitution of variables for variables (*renaming*, *exchange*, and also *weakening* and *contraction* if non-linear)
(as was done for separation logic in Bob Atkey's PhD thesis)

- Not syntax-directed: Coherence theorem
- First time linear logic is treated in this way: σ allows weakening and contraction on types !A
A quotient for structural rules (! is delicate)

Depolarisation

Equivalence between three properties:

1. Cuts associate
2. Evaluation order is unimportant (unrestricted β -reduction)
3. The effect adjunction is idempotent

“Depolarisation”

Suggests new approaches to intuitionistic logic and linear logic
(see the works of Melliès)

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1. Linear Call-by-Push-Value: effect adjunction + resource adjunction
2. Decomposition of cartesian models into linear ones (Girard translation)
3. Simple and general technique for calculi (in Barendregt-style)
4. Connection with intuitionistic (linear) logic via polarisation
5. Characterisation of depolarisation
 - Do not conflate effects and resources
 - Not a naive & one-to-one correspondence between algebraic models and calculi
 - Start with abstract-machine-like calculi (λ -calculus is syntactic sugar)
 - Do not represent structural rules explicitly (terms should provide a quotient modulo structural rules/the monoidal laws)

Thank you

Questions?