1 Focusing

$$
\begin{aligned}
(\Gamma \vdash A \multimap B) &\succ \{(\Gamma, A \vdash B)\} \\
(\Gamma, A \otimes B, \Gamma' \vdash \Delta) &\succ \{(\Gamma, A, B, \Gamma' \vdash \Delta)\} \\
(\Gamma, \mathbf{1}, \Gamma' \vdash \Delta) &\succ \{(\Gamma, \Gamma' \vdash \Delta)\} \\
(\Gamma \vdash A \mathbin{\&} B) &\succ \{(\Gamma \vdash A), (\Gamma \vdash B)\} \\
(\Gamma, A \oplus B, \Gamma' \vdash \Delta) &\succ \{(\Gamma, A, \Gamma' \vdash \Delta), (\Gamma, B, \Gamma' \vdash \Delta)\} \\
(\Gamma \vdash \top) &\succ \emptyset \\
(\Gamma, \mathbf{0}, \Gamma' \vdash \Delta) &\succ \emptyset
\end{aligned}
$$

$$\mathcal{M} \uplus \{\Phi\} \succ_{\mathrm{m}} \mathcal{M} \uplus \mathcal{N} \text{ whenever } \Phi \succ \mathcal{N}$$

Figure 1: Inversion relation between multisets of sequents

$$
\begin{aligned}
c : (\Gamma, z : A \otimes B, \Gamma' \vdash \Delta) &\succ \{\langle x \otimes y \,\|\, \tilde{\mu} z.c \rangle : (\Gamma, x : A, y : B, \Gamma' \vdash \Delta)\} \\
c : (\Gamma, z : A \oplus B, \Gamma' \vdash \Delta) &\succ \{\langle \iota_1(x) \,\|\, \tilde{\mu} z.c \rangle : (\Gamma, x : A, \Gamma' \vdash \Delta), \langle \iota_2(y) \,\|\, \tilde{\mu} z.c \rangle : (\Gamma, y : B, \Gamma' \vdash \Delta)\} \\
c : (\Gamma, x : \mathbf{1}, \Gamma' \vdash \Delta) &\succ \{\langle () \,\|\, \tilde{\mu} x.c \rangle : (\Gamma, \Gamma' \vdash \Delta)\} \\
c : (\Gamma \vdash \beta : A \multimap B) &\succ \{\langle \mu\beta.c \,\|\, x \cdot \alpha \rangle : (\Gamma, x : A \vdash \alpha : B)\} \\
c : (\Gamma \vdash \gamma : A \mathbin{\&} B) &\succ \{\langle \mu\gamma.c \,\|\, \pi_1 \cdot \alpha \rangle : (\Gamma \vdash \alpha : A), \langle \mu\gamma.c \,\|\, \pi_2 \cdot \beta \rangle : (\Gamma \vdash \beta : B)\} \\
c : (\Gamma, x : \mathbf{0}, \Gamma' \vdash \Delta) &\succ \emptyset \\
c : (\Gamma \vdash \alpha : \top) &\succ \emptyset
\end{aligned}
$$

Figure 2: Inversion step of a command

$$\frac{c : (\Gamma, x : A, y : B, \Gamma' \vdash \Delta)}{\langle z \parallel \tilde{\mu}(x{\otimes}y).c \rangle : (\Gamma, z : A \otimes B, \Gamma' \vdash \Delta)} \qquad \frac{c : (\Gamma, x : A \vdash \alpha : B)}{\langle \mu(x{\cdot}\alpha).c \parallel \beta \rangle : (\Gamma \vdash \beta : A \rightarrow B)}$$

$$\frac{c : (\Gamma, x : A, \Gamma' \vdash \Delta) \quad c' : (\Gamma, y : B, \Gamma' \vdash \Delta)}{\langle z \parallel \tilde{\mu}[x.c \,|\, y.c'] \rangle : (\Gamma, z : A \oplus B, \Gamma' \vdash \Delta)} \qquad \frac{c : (\Gamma \vdash \alpha : A) \quad c' : (\Gamma \vdash \beta : B)}{\langle \mu{<}\alpha.c ; \beta.c'{>} \parallel \gamma \rangle : (\Gamma \vdash \gamma : A \,\&\, B)}$$

$$\frac{c : (\Gamma, \Gamma' \vdash \Delta)}{\langle x \parallel \tilde{\mu}().c \rangle : (\Gamma, x : \mathbf{1}, \Gamma' \vdash \Delta)}$$

$$\frac{}{\langle x \parallel \tilde{\mu}[]_{\Gamma,\Gamma',\Delta} \rangle : (\Gamma, x : \mathbf{0}, \Gamma' \vdash \Delta)} \qquad \frac{}{\langle \mu{<>}_{\Gamma} \parallel \alpha \rangle : (\Gamma \vdash \alpha : \top)}$$

Figure 3: Inversion phase

$$\frac{c : (\Gamma \vdash \alpha : N)}{\Gamma \vdash \mu\alpha^{\ominus}.c : N \mid} \qquad \frac{c : (\Gamma, x : P, \Gamma' \vdash \Delta)}{\Gamma, \Gamma' \mid \tilde{\mu}x^{+}.c : P \vdash \Delta}$$

$$\frac{}{\Gamma, x : X^{+}, \Gamma' \vdash x : X^{+} \mid} \qquad \frac{}{\Gamma \mid \alpha : X^{\ominus} \vdash \alpha : X^{\ominus}}$$

$$\frac{\Gamma \vdash V : A \mid \quad \Gamma \vdash W : B \mid}{\Gamma \vdash V{\otimes}W : A \otimes B \mid} \qquad \frac{\Gamma \vdash V : A \mid \quad \Gamma \mid S : B \vdash \Delta}{\Gamma \mid V{\cdot}S : A \rightarrow B \vdash \Delta}$$

$$\frac{}{\Gamma \vdash () : \mathbf{1} \mid} \qquad \frac{\Gamma \vdash V : A_i \mid}{\Gamma \vdash \iota_i(V) : A_1 \oplus A_2 \mid} \qquad \frac{\Gamma \mid S : A_i \vdash \Delta}{\Gamma \mid \pi_i{\cdot}S : A_1 \,\&\, A_2 \vdash \Delta}$$

$$\frac{\Gamma \vdash V : P \mid}{\langle V \parallel \alpha \rangle^{+} : (\Gamma \vdash \alpha : P)} \qquad \frac{\Gamma, x : N \mid S : N \vdash \Delta}{\langle x \parallel S \rangle^{\ominus} : (\Gamma, x : N \vdash \Delta)}$$

Figure 4: Focusing phase in $\mathbf{LJ}_p^{\eta}$

2 LL

$$
\begin{array}{llll}
\text{Formulae:} & A, B, A_\varepsilon & ::= & P \mid N \\
\text{Positives:} & P, Q, A_+ & ::= & X \mid \mathbf{1} \mid A \otimes B \mid A \oplus B \mid \mathbf{0} \\
\text{Negatives:} & N, M, A_- & ::= & \overline{X} \mid \bot \mid A \parr B \mid A \,\&\, B \mid \bot
\end{array}
$$

(a) Formulae

$$
\begin{array}{ccccc}
\overline{A \otimes B} \stackrel{\text{def}}{=} \overline{A} \parr \overline{B} & \overline{\mathbf{1}} \stackrel{\text{def}}{=} \bot & \overline{A \oplus B} \stackrel{\text{def}}{=} \overline{A} \,\&\, \overline{B} & \overline{\mathbf{0}} \stackrel{\text{def}}{=} \top & \overline{(X)} \stackrel{\text{def}}{=} \overline{X} \\
\overline{A \parr B} \stackrel{\text{def}}{=} \overline{A} \otimes \overline{B} & \overline{\bot} \stackrel{\text{def}}{=} \mathbf{1} & \overline{A \,\&\, B} \stackrel{\text{def}}{=} \overline{A} \oplus \overline{B} & \overline{\top} \stackrel{\text{def}}{=} \mathbf{0} & \overline{\overline{X}} \stackrel{\text{def}}{=} X
\end{array}
$$

(b) Dual of a formula

Figure 5: **MALL** formulae and duality

$$
\begin{array}{llllllllll}
 & & & & & \mathbf{1}/\bot & & \otimes/\parr & & \oplus/\& \,(i \in \{1,2\}) & & \mathbf{0}/\top \\
\text{(Co)Values:} & V, W & ::= & x, y, \ldots & \mid \mu x^{\ominus}.c & \mid & () & \mid & V \otimes W & \mid & \iota_i(V) & \mid & \mu[V] \\
 & & & & & \mid & \mu().c & \mid & \mu(x \parr y).c & \mid & \mu[x.c \mid y.c'] \\
\text{(Co)Expressions:} & t, u & ::= & V & \mid \mu x^{+}.c \\
\text{Commands:} & c & ::= & \langle t \parallel V \rangle & (\stackrel{\text{not.}}{=} \langle V \parallel t \rangle)
\end{array}
$$

(a) Terms

$$
\begin{array}{rccl}
(R\mu^{\varepsilon}): & \langle \mu x^{+}.c \parallel V \rangle & \rhd_R & c[V/x] \\
(R\mu^{\varepsilon}): & \langle V \parallel \mu x^{\ominus}.c \rangle & \rhd_R & c[V/x] \\
(R1/\bot): & \langle () \parallel \mu().c \rangle & \rhd_R & c \\
(R\otimes/\parr): & \langle V \otimes W \parallel \mu(x \parr y).c \rangle & \rhd_R & c[V/x, W/y] \\
(R\oplus/\&): & \langle \iota_i(V) \parallel \mu[x_1.c_1 \mid x_2.c_2] \rangle & \rhd_{\mathrm{R}} & c_i[V/x_i] \\
(\text{no rule } \boldsymbol{R0}/\top) & & &
\end{array}
$$

(b) Reduction rules

$$
\begin{array}{rccl}
(E\mu^{+}): & \mu x^{+}.\langle t \parallel x \rangle & \rhd_E & t \\
(E\mu^{\ominus}): & \mu x^{\ominus}.\langle x \parallel V \rangle & \rhd_E & V \\
(E1/\bot): & \mu().\langle () \parallel V \rangle & \rhd_E & V \\
(E\otimes/\parr): & \mu(x \parr y).\langle x \otimes y \parallel V \rangle & \rhd_E & V \\
(E\oplus/\&): & \mu\big[x.\langle \iota_1(x) \parallel V \rangle \big| y.\langle \iota_2(y) \parallel V \rangle\big] & \rhd_{\mathrm{E}} & V \\
(E\mathbf{0}/\top)^{\lightning}: & \mu[x_1 \otimes \cdots \otimes x_n] & \rhd_{\mathrm{E}} & V
\end{array}
$$

(c) Extensionality rules ($\lightning$: the η rule for $\mathbf{0}/\top$ is meaningless without a valid typing judgement)

Figure 6: $\mathbf{MALL}_p^{\eta}$: calculus

- **$\Gamma, \Delta \ldots$** are maps from a finite set of variables to types provided with total orders on their domain, notation $(x_1 : A_1, \ldots, x_n : A_n)$.
- Concatenation (Γ, Δ) is defined when the domains of Γ and Δ are disjoint.
- **$\Sigma^*(\Gamma; \Gamma')$** is the set of bijections $\sigma : \mathrm{dom}\,\Gamma \to \mathrm{dom}\,\Gamma'$ satisfying $\Gamma'(\sigma(x)) = \Gamma(x)$ for all $x \in \mathrm{dom}\,\Gamma$.
- Judgements are: $c : (\vdash \Gamma) \qquad \vdash t : A \mid \Gamma$

(a) Judgements

$$\frac{}{\vdash x : A \mid x : \overline{A}}\ (\mathrm{ax}) \qquad \frac{c : (\vdash x : A_\varepsilon, \Gamma)}{\vdash \mu x^\varepsilon.c : A_\varepsilon \mid \Gamma}\ (\mu^\varepsilon)$$

$$\frac{\vdash t : A \mid \Gamma \qquad \vdash u : \overline{A} \mid \Delta}{\langle t \parallel u \rangle : (\vdash \Gamma, \Delta)}\ (\mathrm{cut})$$

(b) Identity

$$\frac{\vdash t : A \mid \Gamma}{\vdash t[\sigma] : A \mid \Gamma'}\ (\vdash \sigma) \qquad \frac{c : (\vdash \Gamma)}{c[\sigma] : (\vdash \Gamma')}\ (\sigma)$$

(c) Structure — $\sigma \in \Sigma^*(\Gamma; \Gamma')$

$$\frac{\vdash V : A \mid \Gamma \qquad \vdash W : B \mid \Delta}{\vdash V \otimes W : A \otimes B \mid \Gamma, \Delta}\ (\otimes^f) \qquad \frac{}{\vdash () : \mathbf{1} \mid}\ (\mathbf{1})$$

$$\frac{c : (\vdash x : A, y : B, \Gamma)}{\vdash \mu(x \parr y).c : A \parr B \mid \Gamma}\ (\parr) \qquad \frac{c : (\vdash \Gamma)}{\Gamma \vdash \mu().c : \bot \mid \Delta}\ (\bot)$$

$$\frac{c : (\vdash x : A, \Gamma) \qquad c' : (\vdash y : B, \Gamma)}{\vdash \mu[x.c \mid y.c'] : A \,\&\, B \mid}\ (\&) \qquad \frac{\Gamma \vdash V : A \mid}{\Gamma \vdash \mu[V] : \top \mid}\ (\top^f)$$

$$\frac{\vdash V : A_i \mid \Gamma}{\vdash \iota_i(V) : A_1 \oplus A_2 \mid \Gamma}\ (\oplus_i^f) \qquad \text{(no rule for } \mathbf{0}\text{)}$$

(d) Logic

Figure 7: $\mathbf{MALL}_p^\eta$: simple types

$$\frac{\vdash t : A \mid \Gamma \qquad \vdash u : B \mid \Delta}{\vdash t \otimes u : A \otimes B \mid \Gamma, \Delta}\ (\vdash \otimes) \qquad t_{\varepsilon_1} \otimes u_{\varepsilon_2} \stackrel{\text{def}}{=} \mu x^+.\Big\langle t \,\Big\|\, \mu y^{\varepsilon_1}.\big\langle u \,\big\|\, \mu z^{\varepsilon_2}.\langle y \otimes z \parallel x \rangle \big\rangle \Big\rangle$$

$$\frac{\Gamma \vdash t : (A_i)_\varepsilon \mid}{\Gamma \vdash \iota_i(t) : A_1 \oplus A_2 \mid}\ (\vdash \oplus_i) \qquad \iota_i(t_\varepsilon) \stackrel{\text{def}}{=} \mu x^+.\big\langle t \,\big\|\, \mu y.\langle \iota_i(y) \parallel \alpha \rangle \big\rangle$$

$$\frac{}{\Gamma \vdash \mu[]_\Gamma : \top \mid}\ (\vdash \top) \qquad \mu[]_\Gamma \stackrel{\text{def}}{=} \mu[x_1 \otimes \cdots \otimes x_n] \text{ pour } \{x_1, \ldots, x_n\} = \mathrm{dom}\,\Gamma$$

Figure 8: Remaining rules of **MALL** (without value restriction)

$$\text{Positives:} \quad P, Q, A_+ \quad ::= \quad \dots \mid\, !A$$
$$\text{Negatives:} \quad N, M, A_- \quad ::= \quad \dots \mid\, ?A$$

(a) Formulae

$$\overline{!A} \stackrel{\text{def}}{=} ?\overline{A} \qquad \overline{?A} \stackrel{\text{def}}{=} !\overline{A}$$

(b) Dual of a formula

$$\text{(Co)Values:} \quad V, W \quad ::= \quad \dots \quad \mid \quad ?V \quad \mid \quad \mu!x.c$$

(c) Terms

$$(R!/?) : \quad \langle \mu!x.c \parallel ?V \rangle \quad \rhd_R \quad c[V/x]$$

(d) Reduction rules

$$(E!/?) : \quad \mu!x.\langle V \parallel ?x \rangle \quad \rhd_E \quad V$$

(e) Extensionality rules

$$\frac{\vdash t : A \mid \Gamma}{\vdash t[\sigma] : A \mid \Gamma'} \ (\vdash \sigma) \qquad \frac{c : (\vdash \Gamma)}{c[\sigma] : (\vdash \Gamma')} \ (\sigma)$$

(f) Structure for $\sigma \in \Sigma^!(\Gamma; \Gamma')$: the subset of $\Sigma(\Gamma; \Gamma')$ of maps which are bijective on variables not of the form $!A$.

$$\frac{c : (\vdash x : A, !\Gamma)}{\vdash \mu!x.c : !A \mid !\Gamma} \ \text{(prom.)} \qquad \frac{\vdash V : A \mid \Gamma}{\vdash ?V : ?A \mid \Gamma} \ (\text{der.}^f)$$

(g) Logic — where $!(x_1 : A_1, \dots, x_n : A_n)$ stands for the typing context $(x_1 : !A_1, \dots, x_n : !A_n)$.

$$\frac{\vdash t : A \mid ?\Gamma}{\vdash !t : !A \mid ?\Gamma} \ \text{(der.)} \qquad ?t_\varepsilon \quad \stackrel{\text{def}}{=} \quad \mu x^\ominus.\big\langle \mu y^\varepsilon.\langle y \parallel ?\alpha \rangle \,\big\|\, t \big\rangle$$

(h) Remaining rule of **LL** (without value restriction)

Figure 9: $\mathbf{LL}_p^\eta = \mathbf{MALL}_p^\eta +$ the above